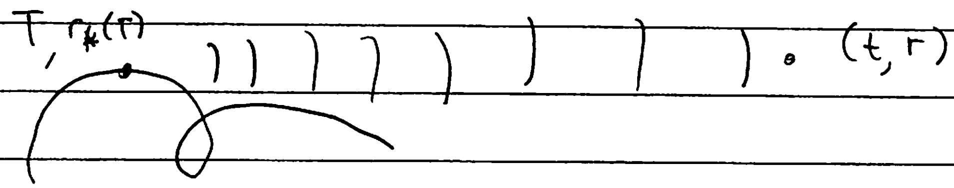
Last Time

- Discussed Radiation



$$-\Box \varphi = \rho$$

$$-\Box A = J/c$$

Use the Grn function of wave-eqn to solve

From Here we derived the Lienard-Wiechert Potentials. In the far field these are

$$\varphi = \frac{e}{4\pi r} \frac{1}{(1 - \vec{n}\cdot\vec{\beta})}$$

$$T = t - \frac{r}{c} + \frac{\vec{n}}{c} \cdot (\vec{r} - \vec{r}_*(T))$$

$$\vec{A} = \frac{e}{4\pi r} \frac{\vec{v}(T)/c}{(1 - n\cdot\beta(T))}$$

Then solved for fields:

$$\vec{E}_{rad} = n \times n \times \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \frac{e}{4\pi r c^2} \frac{n \times (\vec{n} - \vec{\beta}) \times \vec{a}}{(1 - n\cdot\beta)^3}$$

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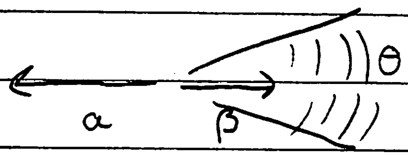
Studied the case for $a \parallel \beta$



$$\frac{dW}{dT\,d\Omega} = \frac{dW}{dt\,d\Omega}\frac{dt}{dT} = c\,|rE_{rad}|^2 (1 - n\cdot\beta)$$

Found

$$\frac{dW}{dT\,d\Omega} = \frac{e^2}{16\pi^2 c^3} \frac{|n \times (n-\beta) \times a|^2}{(1-n\cdot\beta)^5}$$

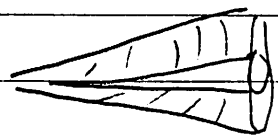
Then we noted that the collinear factor

$$\frac{\partial T}{\partial t} = \frac{1}{(1-n\cdot\beta)}$$ = scale factor between formation time ticks ΔT and observation time Δt ticks

Is strongly enhanced for large γ ($\sim\infty$) and small θ (~ 0), but $\gamma\theta \approx 1$

$$\frac{1}{(1-n\cdot\beta)} \simeq \frac{2\gamma^2}{(1+(\gamma\theta)^2)}$$ (prove me!)

This means that radiation is concentrated in a cone of $\theta \sim \frac{1}{\gamma}$



(Though directly forward there is no radiation)

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Then for $a_{\parallel}$ to β found

$$n \times n \times a = a \sin\theta \simeq a\theta$$

So

$$\frac{dW}{dT\,d\Omega} \sim \frac{e^2 a^2}{c^3} \gamma^8 \frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^5} \qquad d\Omega \simeq 2\pi\theta\, d\theta$$

$$\frac{dW}{dT} \sim \frac{e^2 a^2}{c^3} \gamma^6 \frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)} (\gamma\theta)\, d(\gamma\theta)$$

So

$$\frac{dW}{dT} \sim \frac{e^2 a^2}{c^3} \gamma^6$$

More Precisely Showed

$$\frac{dW}{dT} = \frac{e^2}{4\pi} \frac{2}{3} \frac{\gamma^6}{c^3} \left[a_{\parallel}^2 + \frac{a_{\perp}^2}{\gamma^2} \right]$$

Lienard Wiechert 1898

Analysis of Lienard-Wiechert Result

$$A^{\mu} = \frac{d^2 x^{\mu}}{d\tau^2} = \text{propper acceleration analyzed in homework}$$

In LRF of particle (LRF = local rest frame

$$A^{\mu} = \begin{pmatrix} 0 \\ \alpha_{\parallel} \\ \alpha_{\perp} \end{pmatrix} \qquad A^{\mu} A_{\mu} = \alpha_{\parallel}^2 + a_{\perp}^2$$

Then hmwrk was to show

$$\gamma^3 a_{\parallel} = \alpha_{\parallel} \quad \text{and} \quad \gamma^2 a_{\perp} = \alpha_{\perp}$$

Then

$$A^{\mu} A_{\mu} = \gamma^6 \left[a_{\parallel}^2 + \frac{a_{\perp}^2}{\gamma^2} \right]$$

See prf at end of lecture

So

$$\frac{dW}{dT} = \frac{e^2}{4\pi} \frac{2}{3} A^{\mu} A_{\mu}$$

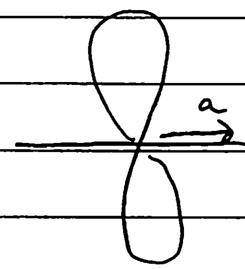
Total Power (Pure Thinking)

In retrospect could "guess" this result

Look at the emission in rest frame of particle

$= A^\mu A_\mu$ in rest frame

energy emitted → $\Delta E = \frac{e^2}{4\pi} \frac{2}{3c^2} a^2 \Delta t$



momentum emitted → $\Delta \vec{P} = 0$ ← Since radiation is emitted symmetricall and transverse to beam

$$\Delta t = \Delta t$$

$$\Delta x = 0$$

Then under boost

$$\underline{\Delta E} = \gamma \, \Delta E$$

$$\begin{pmatrix} \underline{\Delta t} \\ \underline{\Delta P} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \Delta E \\ \Delta P \end{pmatrix}$$

$$\underline{\Delta t} = \gamma \, \Delta t$$

And

$$\text{total power} = \frac{\underline{\Delta E}}{\underline{\Delta t}} = \frac{\Delta E}{\Delta t} = \text{invariant}$$

$$= \frac{e^2}{4\pi} \frac{2}{3c^3} \underbrace{A^\mu A_\mu}_{\text{true in all}}$$

Linear vs. Circular Acclerators

In general, since $p^\mu = m u^\mu$, and $A^\mu = du^\mu/d\tau$

$$\frac{dW}{d\tau} = \frac{e^2}{4\pi}\frac{2}{3} A^\mu A_\mu = \frac{e^2}{4\pi}\frac{2}{3}\frac{1}{m^2c^3}\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}$$

- Then for a linear accelerator where $d\vec{p}/dt$ is parallel to v

$$\frac{d\vec{p}}{d\tau} = \gamma\frac{d\vec{p}}{dt} \qquad \frac{dp^0}{d\tau} = \frac{d\sqrt{p^2+m^2}/c}{d\tau} = \frac{v}{c}\frac{dp}{d\tau} = \frac{\gamma v}{c}\frac{d\vec{p}}{dt}$$

So

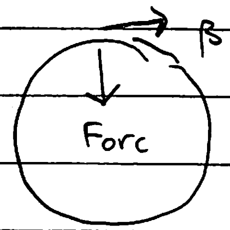
$$\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = -\left(\frac{dp^0}{d\tau}\right)^2 + \left(\frac{dp}{d\tau}\right)^2 = \left(\frac{d\vec{p}}{dt}\right)^2$$

So that the radiated energy grows with the applied force squared

$$\frac{dW}{d\tau} = \frac{e^2}{4\pi}\frac{2}{3m^2c^3}\left(\frac{d\vec{p}}{dt}\right)^2$$

and is independent of γ

• By contrast for a circular accelerator where $\frac{d\vec{p}}{dt}$ is perpendicular to $\vec{v}$. We have that:



$$\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = \frac{d\vec{p}_\perp}{d\tau}\cdot\frac{d\vec{p}_\perp}{d\tau} = \gamma^2\left(\frac{d\vec{p}_\perp}{dt}\right)^2$$

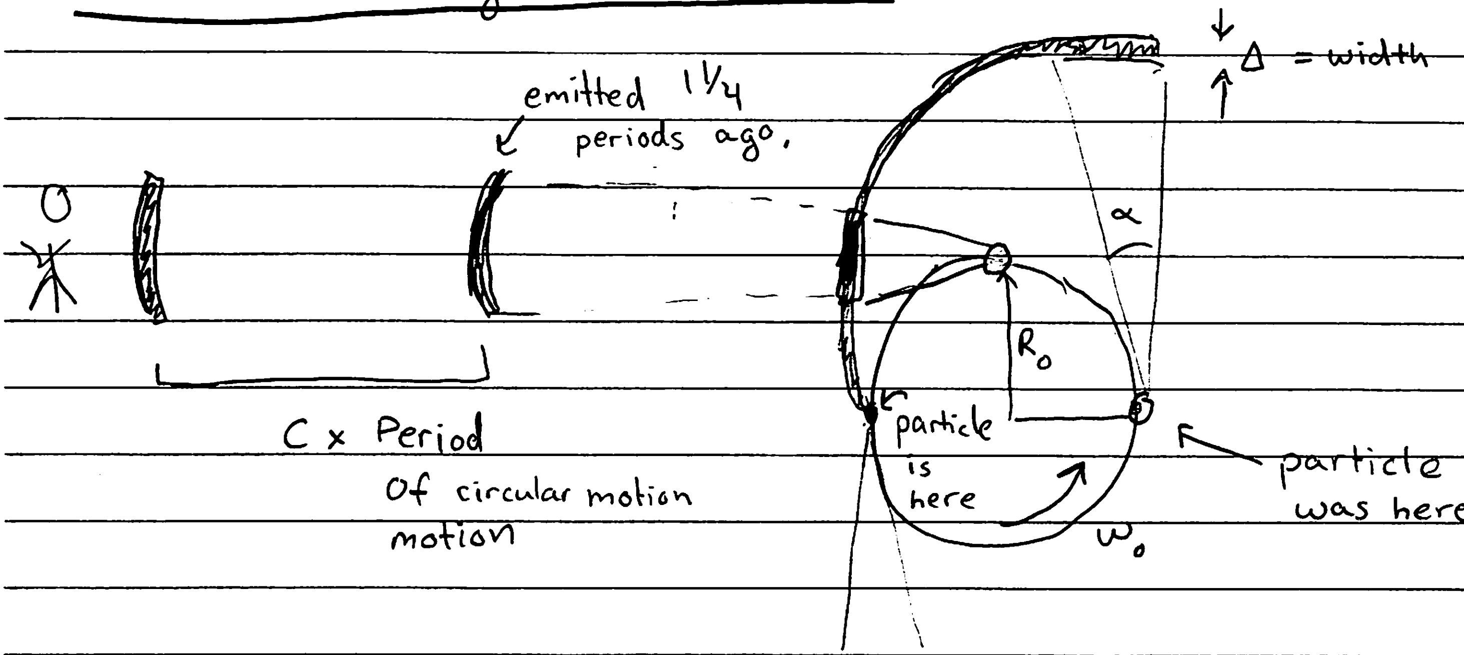
We have that

$$\boxed{\frac{dW}{dT} = \frac{e^2}{4\pi}\frac{2}{3\, m^2c^3}\gamma^2\left(\frac{d\vec{p}_\perp}{dt}\right)^2}$$

So the radiated power grows as γ^2!!

This is becoming prohibitive at colliders today, and is a big reason for research into Linear accelerators

Radiation During Circular Motion (Synchrotron Radiation)



- Every period the strobelight of the radiation cone points in your direction.

- The pulses of light are short in duration: $\Delta t \sim \Delta / c$ the cone is narrow $\alpha \sim \frac{1}{\gamma}$ (and because of the difference between the formation and observation times ... but that we will discuss below)

- The observer sees a pulse every period:

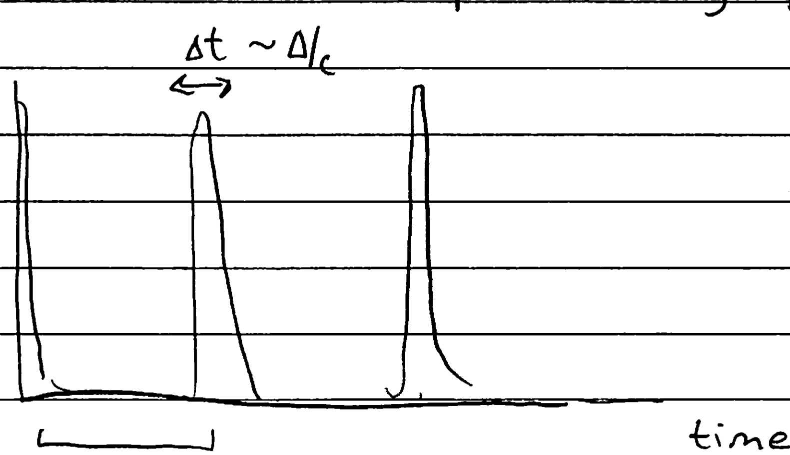


Figure Credit: Christina Athansion et al, arXiv 1001.3880

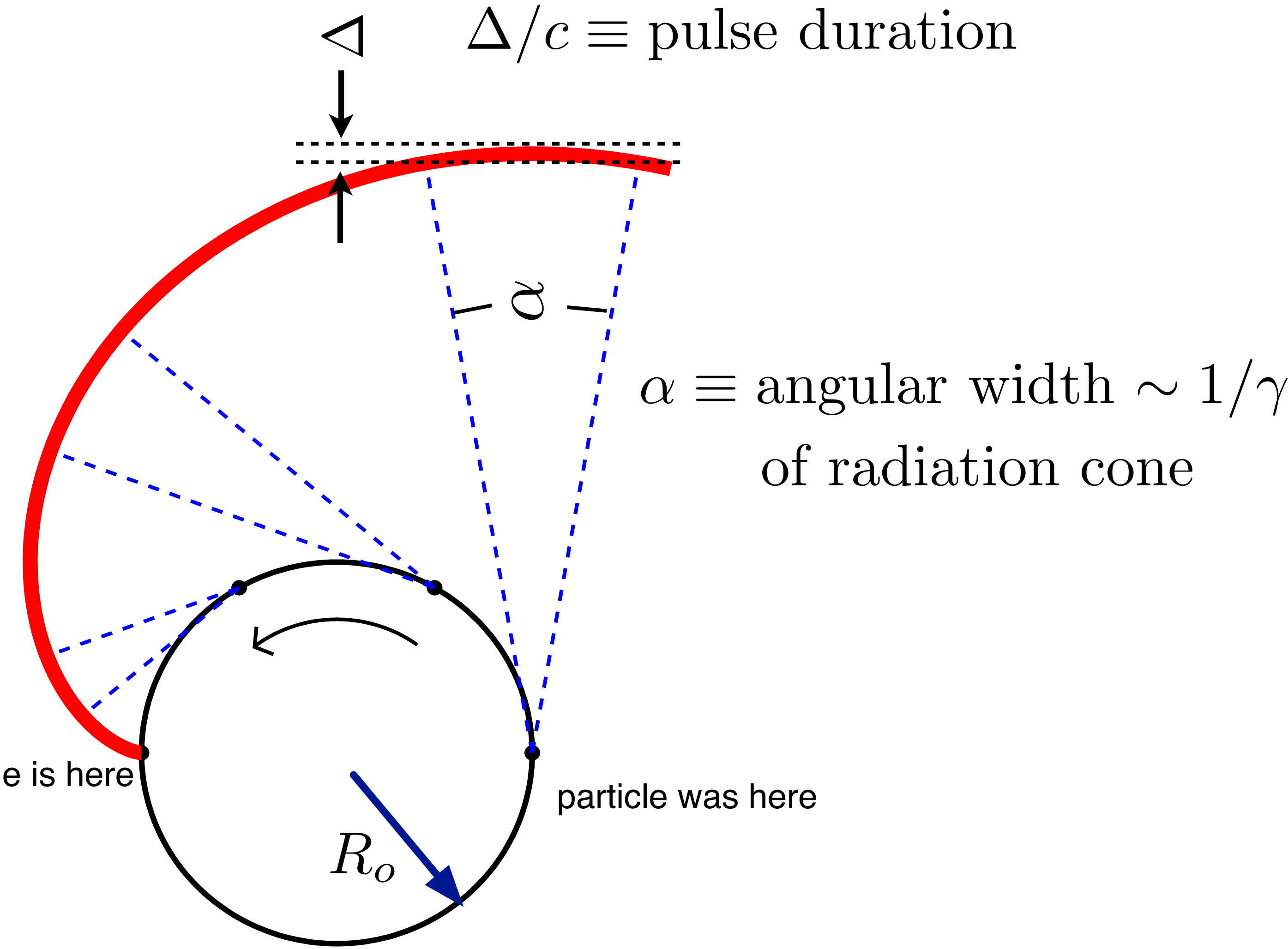
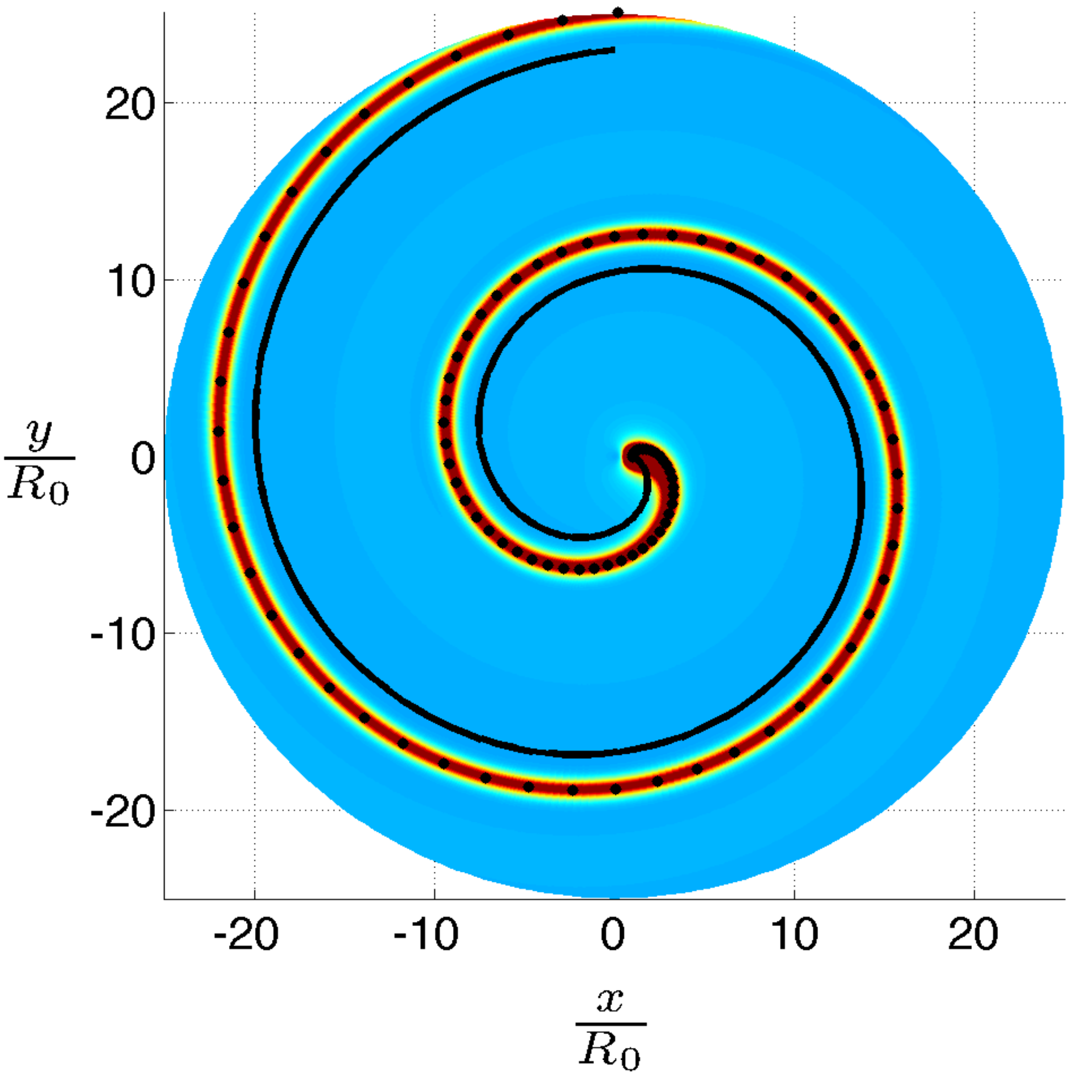


Figure crédit: Christina Athanasion et al, arXiv:1001.3880



Basic Uses of Synchrotron Radiation

- Since the pulse is very narrow in time it contains a wide range of fourier frequencies

$$\Delta \omega \sim \frac{1}{\Delta t}$$

We should compute the pulse shape, look at its fourier transform and compute the power in each band.

- The light is quite intense
- Both of features are highly desirable

Estimate of The Frequency Width

The frequency width is inversely related to the time width, Δt

$$\Delta \omega \sim \frac{1}{\Delta t}$$

Before Starting Definitions:

$\alpha \equiv$ angular width of cone, $\alpha \sim 1/\gamma$

$\delta t \equiv \Delta / c \equiv$ duration of pulse = what we want to estimate

Estimate of $\Delta\omega$ pg. 2

See figures!

① At time T_1 at the saurce (retarded time) the spotlight is starting to point in your angular direction. The leading front is emitted

② The strobelight will point in your direction for a time set by the angular width of radiation cone $\alpha \sim \frac{1}{\gamma}$, and the angular velocity:

$$\Delta T = T_2 - T_1 = \frac{\alpha}{\omega_0} = \frac{R_0 \alpha}{V} \sim \frac{R_0}{\gamma c}$$

$\omega_0 = R_0 / V$

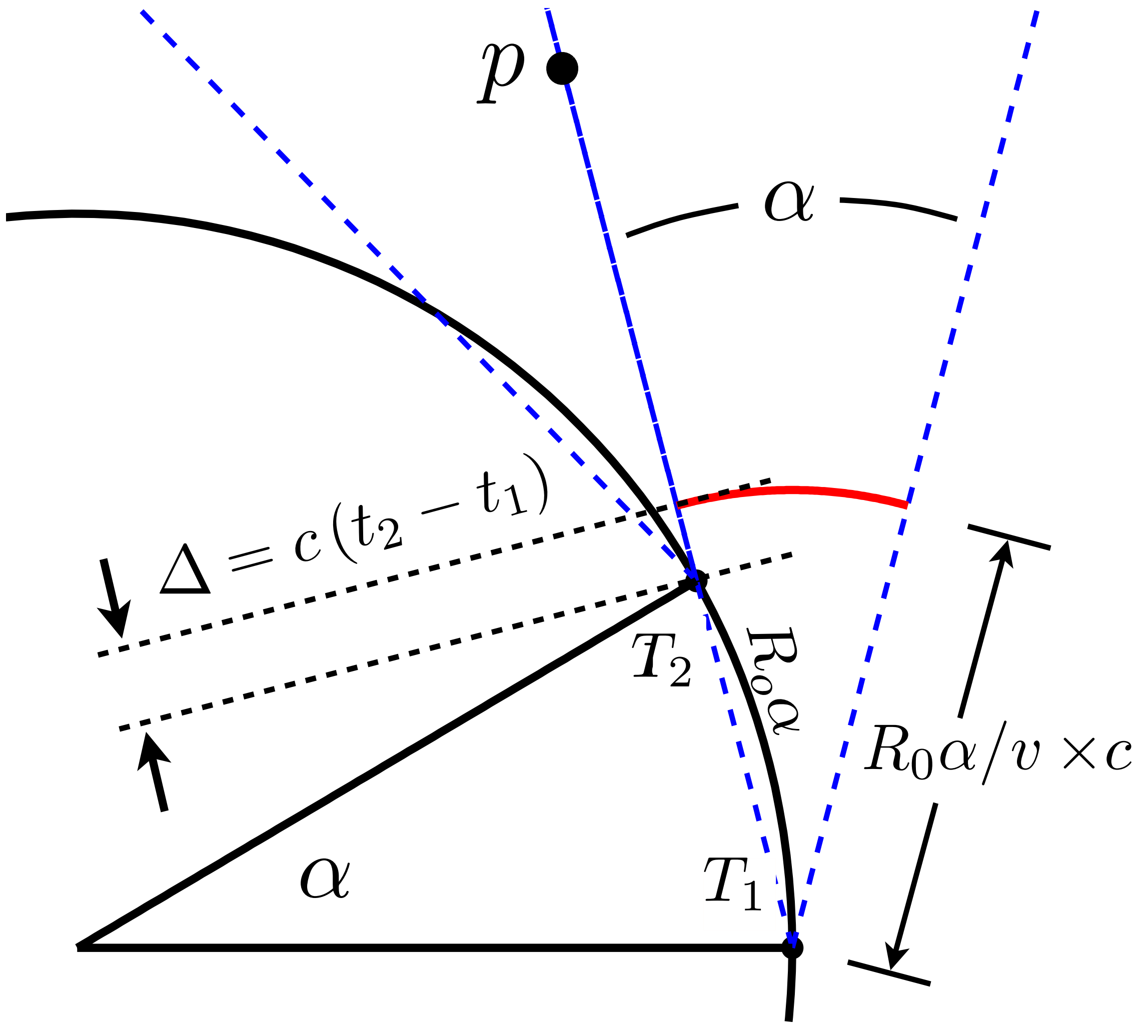
Time at source where the spotlight stops pointing at you

③ Then the kinematics of the emission process says that if the radiation is formed over time ΔT then it is observed to have time scale Δt

$$\Delta t = \frac{\Delta t}{\Delta T} \Delta T = (1 - n\cdot\beta) \frac{R_0}{\gamma V} \sim \frac{(1 + (\gamma\theta)^2)}{2\gamma^2} \frac{R_0}{\gamma c}$$

$$\Delta t \sim \frac{R_0/c}{\gamma^3}$$

Figure Credit: Christina Athanasion et al, arXiv: 1001.3880



Estimate pg. 3

Can also see from geometry

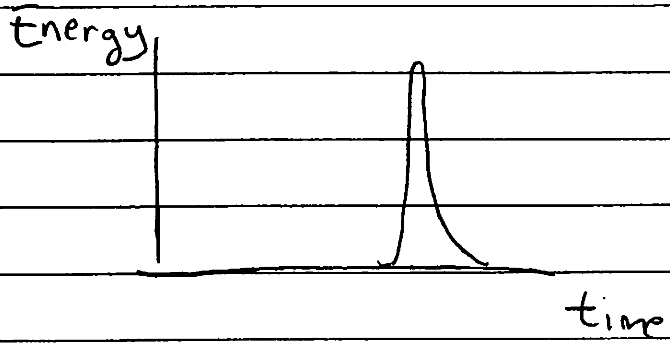
$$c\Delta t = \frac{R_o \alpha c}{v} - R_o \alpha = \underbrace{R_o \alpha}_{\sim \frac{1}{\gamma}} \underbrace{\left(\frac{1}{\beta} - 1\right)}_{1/\gamma^2}$$

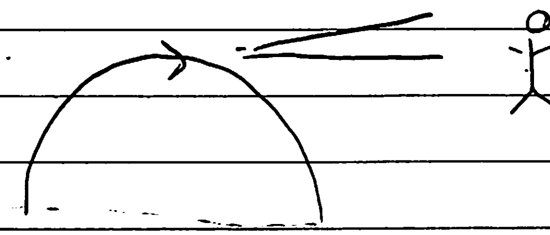
$$\Delta t \sim \frac{R_o / c}{\gamma^3}$$

And

$$\Delta \omega \sim \frac{\gamma^3}{(R_o/c)}$$

The Fourier Spectrum





$$\frac{dW}{dt\,d\Omega} = c\,|r E_{rad}(t)|^2$$ ← energy per observers time

So

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt\; c\,|r E_{rad}(t)|^2\, dt$$

Using Parsevals Thrm (Proved in Homework #1)

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\; c\,|r E_{rad}(\omega)|^2$$

Where

$$E_{rad}(\omega) = \int_{-\infty}^{\infty} e^{+i\omega t}\, E_{rad}(t)$$

$$E_{rad}(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\, E_{rad}(\omega)$$

Fourier Spectrum pg. 2

So we have that

$$2\pi \frac{dW}{d\omega\, d\Omega} = c\left| r\, E_{rad}(\omega) \right|^2$$

The sign of ω is not physically relavant. Since $E(t)$ is real $E(-\omega) = E^*(\omega)$. Thus define (also incorparating a 2π)

$$\frac{dI}{d\omega\, d\Omega} = \frac{c}{2\pi}\left(\left| r E_{rad}(\omega) \right|^2 + \left| r E_{rad}(-\omega) \right|^2 \right)$$

$$\frac{dI}{d\omega\, d\Omega} = \frac{c}{\pi} \left| r E_{rad}(\omega) \right|^2 \quad \text{with } \omega > 0$$

So that

$$\frac{dW}{d\Omega} = \int_0^\infty \frac{dI}{d\omega\, d\Omega}\, d\omega$$

So the number of photons between ω + $(\omega + d\omega$

$$\hbar\omega \frac{dN}{d\omega\, d\Omega}\, d\omega = \frac{dI}{d\omega\, d\Omega}\, d\omega$$