Last Time

- Discussed Radiation

\[ T \equiv \{ \frac{\partial}{\partial t} \} \mathcal{D}(t, \mathbf{r}) \]

\[ \nabla^2 \varphi = \rho \]
\[ \nabla A = \mathbf{J}/c \]

\{ Use the Green function of wave-eqn to solve \}

From here we derived the Lienard-Wiechart Potentials. In the far field these are

\[ \varphi = \frac{e}{4\pi \mathbf{r} \cdot \mathbf{\hat{n}} \cdot \mathbf{\hat{\beta}}} \left( 1 - \mathbf{\hat{n}} \cdot \mathbf{\hat{\beta}} \right) \]

\[ T = t - r + \mathbf{\hat{n}} \cdot t (\mathbf{r} - \mathbf{\hat{r}}(t)) \]

\[ \mathbf{A} = \frac{e}{4\pi \mathbf{r} \cdot \mathbf{\hat{n}} \cdot \mathbf{\hat{\beta}}(t)} \frac{\mathbf{J}(t)}{c} \]

Then solved for fields:

\[ \mathbf{E}_{\text{rad}} = \mathbf{n} \times \mathbf{n} \times \frac{1}{2} \mathbf{\hat{A}} = \frac{e}{c} \frac{\mathbf{n} \times (\mathbf{\hat{n}} \cdot \mathbf{\hat{\beta}}) \times \mathbf{\hat{A}}}{4\pi \mathbf{r} \cdot \mathbf{\hat{n}} \cdot \mathbf{\hat{\beta}}(t)} \]

\[ \frac{1}{1 - \mathbf{n} \cdot \mathbf{\hat{\beta}})^3} \]
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studied the case for \( \alpha \parallel \beta \).

\[
\frac{dw}{dT d\Omega} = \frac{dW}{dt d\Omega} = c \ln E \alpha d l^2 \left( 1 - n \cdot \beta \right)
\]

found

\[
\frac{dW}{dT d\Omega} = \frac{e^2}{16 \pi^2 c^3} \frac{\ln (n-\beta) \times a}{(1 - n \cdot \beta)^5}
\]

Then, we noted that the collinear factor

\[
\frac{\Delta T}{\Delta t (1-\beta)} = \text{scale factor between formation time ticks } \Delta T \text{ and observation time } \Delta t
\]

is strongly enhanced for large \( \beta \) and small \( \Theta \), but

\( \Theta \approx 1 \)

\[
\frac{1}{(1 - n \cdot \beta)} \sim \frac{2 \gamma^2}{(1 + (\gamma \Theta)^2)}
\]

(prove me!)

This means that radiation is concentrated in a cone of \( \Theta \approx \frac{1}{\gamma} \)

(though directly forward there is no radiation)
Then for $\alpha \gamma$ to be found

$$n \times n \times a = a \sin \Theta \sim a \Theta$$

So

$$\frac{dW}{dT d\Omega} \sim \frac{e^2 \alpha^2 \gamma^8 \gamma \Theta^2}{C^3 (1 + \gamma \Theta)^5}$$

$$\Delta \Omega \sim 2 \pi \Theta d\Theta$$

So

$$\frac{dW}{dT} \sim \frac{e^2 \alpha^2 \gamma^6 \gamma \Theta^2 (\gamma \Theta) d(\gamma \Theta)}{C^3 (1 + \gamma \Theta)^2}$$

More precisely showed

$$\frac{dW}{dT} = \frac{e^2}{4 \pi} \frac{2}{3} \frac{\gamma^6}{C^3} \left[ \frac{a_{11}^2 + a_2^2}{\gamma^2} \right]$$

\[ \text{Lienard Wiechert} \ 1898 \]
Analysis of Lienard-Wiechert Result

\[ A^\mu = \frac{d^2 x^\mu}{dt^2} = \text{proper acceleration analyzed in homework} \]

In LRF of particle (LRF = local rest frame)

\[ A^\mu = \begin{pmatrix} 0 \\ \xi_\parallel \\ \xi_\perp \end{pmatrix} \quad A^\mu A_\mu = \alpha_\parallel^2 + \alpha_\perp^2 \]

Then homework was to show

\[ \gamma^3 \alpha_\parallel = \alpha_\parallel \quad \text{and} \quad \gamma^2 \alpha_\perp = \alpha_\perp \]

Then

\[ A^\mu A_\mu = \gamma^6 \left[ \alpha_\parallel^2 + \frac{\alpha_\perp^2}{\gamma^2} \right] \quad \text{of lecture} \]

So

\[ \frac{dW}{dt} = \frac{e^2}{4\pi} \frac{2}{3} A^\mu A_\mu \]
Total Power (Pure Thinking)

In retrospect could "guess" this result.

Look at the emission in rest frame of particle $-A^m A_m$ in rest frame.

Energy $\Rightarrow \Delta E = \frac{e^2}{4\pi} \frac{2}{3c^2} a^2 \Delta t$

Momentum $\Rightarrow \Delta P = 0 \Leftarrow$ Since radiation emitted is emitted symmetrically and transverse to beam.

$\Delta t = \Delta t$

$\Delta x = 0$

Then under boost

$\Delta E = \gamma \Delta E$

$\left( \frac{\Delta t}{\Delta P} \right) = \left( \frac{\gamma \delta \beta}{\delta \beta} \right) (\Delta E)$

$\Delta t = \gamma \Delta t$

And

Total power $= \frac{\Delta E}{\Delta t} = \frac{\Delta E}{\Delta t} = \text{invariant}$

$= \frac{e^2}{4\pi} \frac{2}{3c^2} A^m A_m \rightarrow \text{true in all}$
Linear vs. Circular Accelerators

In general, since $P^\mu = mU^\mu$, and $A^\mu = dU^\mu / dt$

$$\frac{dW}{dt} = \frac{e^2}{4\pi^2} \frac{2}{3} A^\mu A_\mu = \frac{e^2}{4\pi^2} \frac{2}{3} \frac{1}{m^2 c^3} \frac{dp^\mu}{dt} \frac{dp^\mu}{dt}$$

Then for a linear accelerator where $\frac{dp}{dt}$ is parallel to $v$

$$\frac{dp}{dt} = v \frac{dp}{dt} \quad \frac{dp^0}{dt} = \frac{d}{dt} \frac{\sqrt{p^2 + m^2}}{c} = \frac{v}{c} \frac{dp}{dt} - \frac{\gamma v}{c} \frac{dp}{dt}$$

So

$$\frac{dp^\mu}{dt} \frac{dp^\mu}{dt} = -\left(\frac{dp^0}{dt}\right)^2 + \left(\frac{dp}{dt}\right)^2 - \left(\frac{dp^1}{dt}\right)^2$$

So that the radiated energy grows with the applied force squared

$$\frac{dW}{dt} = \frac{e^2}{4\pi^2} \frac{2}{3} \frac{1}{m^2 c^3} \left(\frac{dp}{dt}\right)^2$$

and is independent of $\gamma$
By contrast for a circular accelerator where $d\hat{p}$ is perpendicular to $\frac{d\hat{p}}{dt}$, we have that:

$$\frac{dp^+}{dt} \frac{dp^-}{dt} = \gamma^2 (\frac{d\hat{p}^+}{dt})^2$$

We have that:

$$\frac{dW}{dt} = \frac{e^2 \gamma^2}{4 \pi \varepsilon_0 c^3} \left(\frac{d\hat{p}^+}{dt}\right)^2$$

So the radiated power grows as $\gamma^2$!!

This is becoming prohibitive at colliders today, and is a big reason for research into linear accelerators.
Radiation During Circular Motion (Synchrotron Radiation)

- Every period the stroboslight of the radiation cone points in your direction.

- The pulses of light are short in duration: $\Delta t \sim \Delta t / c$
  the cone is narrow $\alpha \approx 1$ (and because of the difference between the formation and observation times ... but that we will discuss below)

- The observer sees a pulse every period:

  \[
  \Delta t \sim \Delta t / c
  \]
FIG. 2: A sketch of the solutions (2.8). As the quark moves along its circular trajectory, it emits radiation in a narrow cone of angular width $\alpha$. An observer at the point $p$ will see a short pulse of radiation of duration $\Delta/c \equiv \text{pulse duration}$ and of spatial thickness $R/\gamma$ for both the scalar and vector radiation. Clearly, the radial width of the pulse observed at $t$ is a consequence of the fact that the radiation emitted at $t$ will have traveled a distance between the two emission points is approximately $R/\gamma$. The leading edge of the pulse is a periodic function of its argument with period $2\pi$. The cone of radiation emitted at each time propagates as $\Delta \sim 1/\gamma$.

We note that the fact that Eq. (2.10) also implies that a change in the position of the quark corresponding to a shift in its azimuthal angle by $\pi'$ is equivalent to a shift in both $\rho$ and $\theta'$. We now describe the qualitative behavior of the solutions (2.8). As the quark moves along its circular trajectory, an observer at $p$ sees a short pulse of radiation at each time, the radiation emitted at all times in the past forms a spiral, as illustrated by the red spiral sketched in Fig. 2. Because the motion of the quark is circular, the spatial distance between the two emission points is approximately $R/\gamma$.

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quark is beamed in a cone in the direction of the velocity and strong coupling, the energy radiated by the rotating quark therefore reach the following conclusions: at both weak and strong coupling, the energy density propagates at the speed of light without spreading. We have seen in Fig. 5 that at weak coupling the black dots is (2.12), namely the place where the spiral (2.12) marks the maximal energy density red. The spiral curve marked with black dots is (2.12), namely the place where the spiral of energy density in the \( \mathcal{N} = 4 \) SYM theory is beamed in a cone with opening angle \( \theta_c \).

First, as can be seen from Fig. 4, the radiation emitted tangentially, in the direction of the velocity vector of the quark and the fact that the pulse of radiated energy density propagates at the speed of light without spreading. We have already seen that in the angular distribution of power in classical electrodynamics and SYM (2.23a) and strongly coupled SYM (3.73) for velocity scaling as \( \propto \theta_c^2 \Theta^2 \).

By time-averaging, we eliminate all dependence on azimuthal angle but nontrivial dependence on the polar angle \( \theta_c \). From the far-zone expression for the energy density between \( \Theta = 0 \) and \( \Theta = \Theta_c \), it is easy to see that the radiation decreases as \( \Theta \to \Theta_c \). Using the correspondence principle, we may identify this decrease with a length scale in the boundary quantum field theory. We find precise agreement, as illustrated in Fig. 7. At strong coupling, the energy density decreases as \( \Theta \to \Theta_c \).

Intriguingly, the scaling of both radial and angular thickness has \( \Theta \approx \Theta_c \).

Emboldened by the agreement between the geometrical arguments of Section II to the strongly coupled gravitational dual is at the same location. This indicates that, as at weak coupling, strongly coupled synchrotron radiation is beamed in a cone with opening angle \( \theta_c \) in the frame of the rotating quark, corresponding to a length scale \( \Theta \approx \Theta_c \).

Recall that our derivation of (2.12) in Section II was approximated to \( \Theta_c \approx \Theta \), using the approximation to (2.12). It can also be shown that the minima of (3.72) 

\[
\frac{v}{c} \approx 1 - \frac{1}{N^3}.
\]

We now turn to the time-averaged angular distribution of power in classical electrodynamics and SYM (2.23). As is evident from the figure, in both classical electrodynamics and SYM the power is localized at \( \Theta = 0 \) plane, for example as depicted at two velocities in Fig. 4. The radiation emitted by a rotating quark does not broaden.

There may be a holographic interpretation of \( \Theta \approx \Theta_c \).
Basic Uses of Synchrotron Radiation

- Since the pulse is very narrow in time it contains a wide range of Fourier frequencies.

\[ \Delta \omega \sim \frac{1}{\Delta t} \]

- We should compute the pulse shape, look at its Fourier transform, and compute the power in each band."

- The light is quite intense.

- Both of features are highly desirable.

Estimate of The Frequency Width

The frequency width is inversely related to the time width, \( \Delta t \):

\[ \Delta \omega \sim \frac{1}{\Delta t} \]

Before Starting Definitions:

\( \alpha \) = angular width of cone, \( \alpha \sim 1/\delta \)

\( \delta t \equiv \Delta t / c \equiv \text{duration of pulse} \equiv \text{what we want to estimate} \)
Estimate of $\Delta \omega$ pg 2

See figures!

1. At time $T_1$ at the source (retarded time) the spotlight is starting to point in your angular direction. The leading front is emitted.

2. The strobe light will point in your direction for a time set by the angular width of radiation cone $\alpha/\gamma$, and the angular velocity:

$$\Delta T = T_2 - T_1 = \frac{\alpha}{\omega_0} = \frac{R_0}{V} \frac{\alpha}{\gamma c}$$

$$\omega_0 = \frac{R_0}{V}$$

Time at source where the spotlight stops pointing at you.

3. Then the kinematics of the emission process says that if the radiation is formed over time $\Delta t$ then it is observed to have time scale $\Delta t$

$$\Delta t = \Delta t \frac{\Delta T}{\Delta t} = (1 - n \cdot \beta) \frac{R_0}{\delta V} \frac{1 + \alpha (\beta^2)}{V} \frac{R_0}{\gamma c}$$

$$\Delta t \sim \frac{R_0}{\gamma c}$$
trajectory, an observer at the point will see a short pulse of radiation of duration . As we illustrate in Fig. 3, the quark corresponding to a shift in its azimuthal angle .

We now describe the qualitative behavior of the solutions . As the quark moves along its circular trajectory, it emits radiation in a narrow cone of angular width , and of spatial thickness . The leading edge of the pulse observed at is therefore illuminated by a pulse of duration , denoted by the solid red line, has . Moreover, the chordal distance will have traveled a distance . The chordal distance in the direction of the quark's velocity vector, within a cone of angular width , and of duration is therefore . As we illustrate in Fig. 3, the width of the spiral must go to zero as , namely as .

Clearly, the radial width scales like as 

where . We note that the fact that the radiation emitted at all times in the past is centered on a curve in the fact specify the location of the spiral of radiation pre-traversed. The cone of radiation emitted at each time propagates outwards at the speed of light. At any one moment in time, the radiation emitted at all times in the past forms a cone of angular width , and the retarded time will be emitted by the quark at time .

The width of the spiral scales like , and the retarded time . The thickness of the pulse observed at is therefore . From the solutions , we re-use these arguments in the analysis of our strong coupling asymptotics.
Can also see from geometry $\frac{1}{3} \frac{1}{\gamma^2}$

$\Delta t = \frac{R_0 \alpha c}{V} - R_0 \alpha = R_0 \alpha \left( \frac{1}{\beta} - 1 \right)$

$\Delta t \sim \frac{R_0}{1c} \frac{1}{\gamma^3}$

And

$\Delta W \sim \frac{\gamma^3}{(R_0/c)}$
The Fourier Spectrum

energy

\[ \frac{dW}{dt} \propto c |rE(t)|^2 \]  
energy per observers

time

So

\[ \frac{dW}{d\omega} = \int_{-\infty}^{\infty} dt \, c |rE(t)|^2 dt \]

Using Parsevals Thrm. (Proved in Homework #1)

\[ \frac{dW}{d\omega} = \int_{-\infty}^{\infty} \frac{dW}{d\omega} \cdot c |E_{rad}(\omega)|^2 \]

Where

\[ E_{rad}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} E_{rad}(t) \]

\[ E_{rad}(t) = \int_{-\infty}^{\infty} e^{-i\omega t} E_{rad}(\omega) \]
Fourier Spectrum pg. 2

So we have that

\[ 2\pi \frac{dW}{dw d\Omega} = c \left| r E_{\text{rad}}(w) \right|^2 \]

The sign of \( w \) is not physically relevant. Since \( E(t) \) is real \( F(-w) = E^*(w) \). Thus define (also incorporating a \( 2\pi \))

\[ \frac{dT}{dw d\Omega} = \frac{c}{2\pi} \left( \left| r E_{\text{rad}}(w) \right|^2 + \left| r E_{\text{rad}}(-w) \right|^2 \right) \]

\[ \frac{dT}{dw d\Omega} = \frac{c}{\pi} \left| r E_{\text{rad}}(w) \right|^2 \text{ with } w > 0 \]

So that

\[ \frac{dW}{d\Omega} = \int_0^\infty \frac{dT}{dw d\Omega} \, dw \]

So the number of photons between \( w \) and \( w + dw \)

\[ tw \frac{dN}{dw d\Omega} \, dw = \frac{dT}{dw d\Omega} \, dw \]