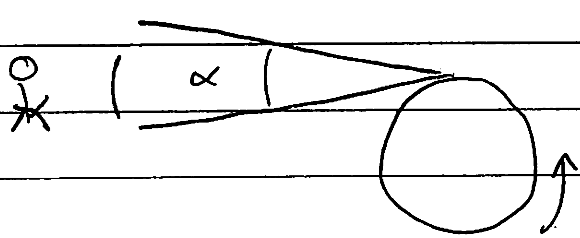
<u>Last Time</u>

- Ultrarelativistic particle radiate in a cone of $\alpha \equiv \Delta\theta \sim 1/\gamma$.

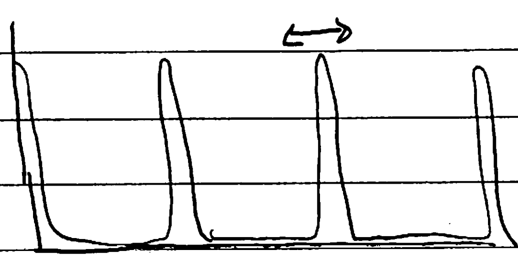
- For a particle travelling in a circle, the total energy radiated per source time is:

$$\frac{dW}{dT} = \frac{e^2}{4\pi}\frac{2}{3c^3}\gamma^4 a_\perp^2$$

- The qualitative features of the radiation pattern are that an oberver sees the light briefly every period when the "strobelight" points in your direction



The observer sees pulses every period. The duration of each pulse is



$$\Delta t \sim \Delta T \frac{\Delta t}{\Delta T}$$

$$\sim \frac{R_0 \Delta\theta}{V} \underbrace{(1-\beta)}_{\sim 1/\gamma^2} \sim \frac{R_0/c}{\gamma^3}$$

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- Now we wish to compute the frequeny spectrum
We expect that the typical frequency is

$$\Delta\omega \sim \frac{1}{\Delta t} \sim \frac{\gamma^3}{R_0/c} = \gamma^3\omega_0$$

We need to find the field, take its fourier transform, and compute the power in each frequency band $|E(\omega)|^2 \Delta\omega$. More precisely we defined:

$$2\pi \frac{dW}{d\omega\, d\Omega} = c\,|rE(\omega)|^2$$

And its positive frequency counterpart

$$\frac{dI}{d\omega\, d\Omega} = \frac{c}{\pi}\,|rE(\omega)|^2 \qquad \omega > 0$$

So,

$$\frac{dW}{d\Omega} = \int_0^\infty \frac{dI}{d\omega\, d\Omega}$$

The Spectrum

$$E(\omega) = \int_{-\infty}^{\infty} dt \; e^{i\omega t} \; E(t)$$

$$= \int_{-\infty}^{\infty} dt \; e^{+i\omega t} \; \frac{q}{4\pi r c^2} \left[\frac{n \times (n-\beta) \times a}{(1 - n\cdot\beta)^3} \right]_{ret}$$

Where all variables β and $a(T)$ are supposed to be evaluated at the retarded time. We want to integrate over retarded time

$$T = t - \frac{r}{c} + \frac{n\cdot r_*}{c} \implies t = T + \frac{r}{c} - \frac{\vec{n}\cdot\vec{r}_*}{c}$$

So using

$$\frac{dt}{dT} = 1 - n\cdot\beta$$

$$E(\omega) = e^{i\omega r/c} \int_{-\infty}^{\infty} dT \; \overbrace{(1-n\cdot\beta)}^{\partial t/\partial T} \; e^{i\omega(T - n\cdot r_o/c)} \; E(T)$$

$$E(\omega) = \frac{q \, e^{i\omega r/c}}{4\pi r c^2} \int_{-\infty}^{\infty} dT \; e^{i\omega(T - n\cdot r_o/c)} \left[\frac{n \times (n-\beta) \times a}{(1 - n\cdot\beta)^2} \right]_{ret}$$

in many ways this form is the simplest. But we can find another form that is often used by recalling that

$$E(T) = n \times n \times \frac{1}{c}\frac{\partial A}{\partial t} = n \times n \times \frac{\partial A}{\partial T}\frac{\partial T}{\partial t}$$

The spectrum pg. 2

So that

$$E(\omega) = e^{i\omega r/c} \int_{-\infty}^{\infty} dT \; e^{i\omega(T - n\cdot r_*/c)} \; \frac{n \times n \times \partial A}{c \;\; \partial T}$$

where, $\vec{A} = \dfrac{q}{4\pi r} \dfrac{\vec{V}(T)/c}{(1 - n\cdot\beta)}$. Integrating by parts,

we have using

$$\frac{d}{dT}(T - n\cdot r_*/c) = 1 - n\cdot\beta$$

We have

$$E(\omega) = \frac{q}{4\pi c} e^{i\omega r/c} (-i\omega) \int_{-\infty}^{\infty} dT \; e^{i\omega(T - n\cdot r_*(T))} \; \vec{n} \times \vec{n} \times \vec{\beta}$$

Or

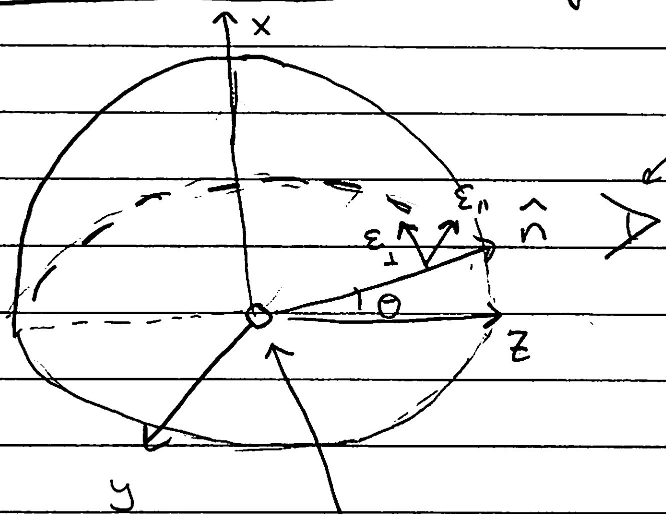
$$\boxed{c\,|E(\omega)|^2 = \frac{q^2}{16\pi^2} \left(\frac{\omega^2}{c^2}\right) \left| \int_{-\infty}^{\infty} dT \; e^{i\omega(T - n\cdot r_*)} \; n \times n \times \beta(T) \right|^2}$$

Eq. (★★★)

This is $2\pi \, dW/d\omega\, d\Omega$. It shows that the electric field is determined by a kind of retarded fourier transform of the transverse current

$$\frac{J_t}{c} = n \times n \times eV/c$$

The Sychrotron Spectrum Using Eq. (★★★)'



detector with incline θ relative to y, z plane

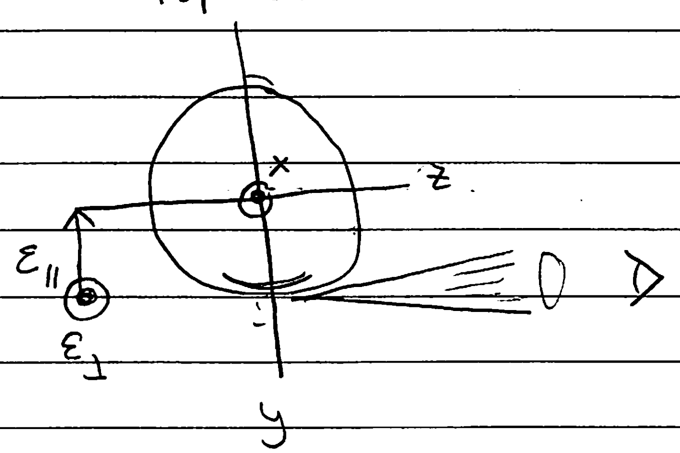
particle going around in a circle here in y, z plane

① Choose the observation direction, so that it lies in x - z plane:

$$\hat{n} = (\sin\theta, 0, \cos\theta)$$

② Take the particle going in the y - z plane:

Top View



$$r_*(T) = R_0 (0, \cos\omega_0 T, \sin\omega_0 T)$$

$$\beta(T) = \frac{v_0}{c} (0, -\sin\omega_0 T, \cos\omega_0 T)$$

we are drawing the motion at T around T=0, and $\theta = 0$

$$\vec{\varepsilon}_\perp = (\cos\theta, 0, \sin\theta), \quad \vec{\varepsilon}_\parallel = -\hat{y}$$

$$\approx \hat{x}$$

Using The Formula for $E(\omega)$, Eq. (★★★)

Need $n \times n \times \beta = -\vec{\beta} + \vec{n}(n \cdot \vec{\beta})$

$$= \frac{V_0}{c}\left(\cos\omega_0 T \cos\theta \sin\theta,\ \sin\omega_0 T,\ -\cos\omega_0 T\,(1-\cos^2\theta)\right)$$

$$\simeq \frac{V_0}{c}\left(\theta,\ \omega_0 T,\ 0\right)$$

see definition of $\mathcal{E}_{\parallel}$ and $\mathcal{E}_{\perp}$.

$$= \theta\,\vec{\mathcal{E}}_{\perp} + -\left(\frac{cT}{R_0}\right)\vec{\mathcal{E}}_{\parallel}$$

Use $V_0 = \omega_0 R_0$.
Use $v \simeq c$

Then we approximate the phase

$$\phi = \omega\left(T - \frac{\vec{n}\cdot\vec{r}_*(T)}{c}\right) = \omega\left(T - \frac{R_0}{c}\sin\omega_0 T\cos\theta\right)$$

Expanding to cubic order with T, θ, small and $1-\beta \simeq 1/2\gamma^2 \simeq (\text{small})^2$ we have

$$\sin\omega_0 T \simeq \omega_0 T + \frac{1}{3}(\omega_0 T)^3, \quad \cos\theta \simeq 1 - \frac{\theta^2}{2}$$

use $V_0 = \omega_0 R_0$

So

$$\phi = \omega T\left(1 - \frac{V_0}{c}\cos\theta\right) + \frac{\omega}{3!}\frac{R_0\omega_0^3 T^3}{c}$$

$$\simeq \frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)T + \frac{c^2}{3R^2}T^3\right]$$

use $\omega_0 \simeq \frac{c}{R}$

and use

$$\left(1 - \frac{V}{c}\cos\theta\right) \simeq \frac{1}{2\gamma^2} + \frac{\theta^2}{2}$$

So now we can evaluate $|E(\omega)|^2$, Eq ($\star\star\star$)

From two pages back

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2}{16\pi^2} \frac{\omega^2}{c} \left| \int_{-\infty}^{\infty} dT \, e^{i\omega (T - \frac{n \cdot r_*(T)}{c})} \vec{n} \times \vec{n} \times \vec{\beta}(T) \right|^2$$

Find

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2}{16\pi^2} \frac{\omega^2}{c} \left| \overbrace{A_\perp \vec{\mathcal{E}}_\perp}^{\propto v_0} + \overbrace{(-A_{\parallel})}^{\propto cT/R} \vec{\mathcal{E}}_\parallel \right|^2$$

Where

$$A_\perp = \theta \int_{-\infty}^{\infty} dT \, e^{\frac{i\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)T + \frac{1}{3}\frac{c^2}{R^2}T^3\right]}$$

$$A_\parallel = \int_{-\infty}^{\infty} dT \, e^{\frac{i\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)T + \frac{1}{3}\frac{c^2}{R^2}T^3\right]} \frac{cT}{R}$$

Now we rescale the retarded time and the frequency

$$x \equiv \frac{cT}{R_0} \frac{1}{(1/\gamma^2+\theta^2)^{1/2}} \quad \text{and} \quad \xi = \frac{\omega R_0/c}{3\gamma^3} \left(1+(\gamma\theta)^2\right)^{3/2}$$

This may seem mysterious but the rescalings are chosen so:

$$\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)T + \frac{1}{3}\frac{c^2 T^3}{R^2}\right] = \frac{3}{2}\xi\left(x + \frac{1}{3}x^3\right)$$

Evaluating Eq ($\star\star\star$) pg. 4

Then

$$A_{\parallel} = \frac{R_0}{c} \underbrace{\left(\frac{1}{\gamma^2} + \theta^2\right)}_{\propto \xi^{2/3}} \underbrace{\int_{-\infty}^{\infty} dx \, e^{i\frac{3}{2}\xi\left(x + \frac{x^3}{3}\right)} x}_{\frac{2}{\sqrt{3}} K_{2/3}(\xi)}$$

$\frac{2}{\sqrt{3}} K_{2/3}(\xi)$ ← airy integral or modified bessel

$$A_{\perp} = \frac{R_0 \theta}{c} \underbrace{\left(\frac{1}{\gamma^2} + \theta^2\right)^{1/2}}_{\propto \xi^{1/3}} \underbrace{\int_{-\infty}^{\infty} dx \, e^{i\frac{3}{2}\xi\left(x + \frac{1}{3}x^3\right)}}_{\frac{2}{\sqrt{3}} K_{1/3}(\xi)}$$

So then

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left[\overbrace{\left(\frac{\omega R_0}{3c\gamma^3}\right)^{2/3}\left(\xi^{2/3} K_{2/3}(\xi)\right)^2}^{\text{parallel polarization contribution}} + \underbrace{\left(\frac{\omega R_0}{3c\gamma^3}\right)^{4/3}\left(\gamma\theta\,\xi^{1/3} K_{1/3}(\xi)\right)^2}_{\text{perpendicular polarization contribution}}\right]$$

We should Analyze this:

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2\gamma^2}{c} \underbrace{F\left(\frac{\omega\gamma^3}{(R_0/c)}, \gamma\theta\right)}_{\text{dimensionless order 1 function}}$$

- The characteristic angle is set by $\gamma\theta \sim 1$ or

$$\theta \sim 1/\gamma \quad \leftarrow \text{this is what we found previously}$$

- The characteristic frequency is set by $\omega\gamma^3/(R_0/c) \sim 1$ or

$$\omega \sim \frac{\gamma^3}{R_0/c}$$

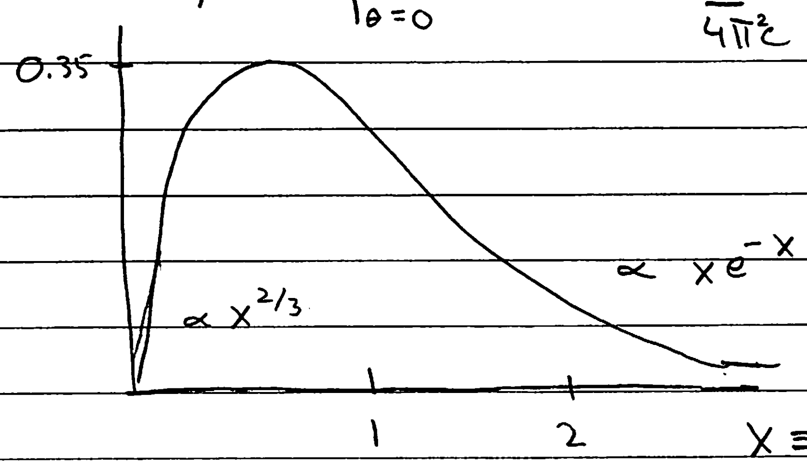
Analysis of Formula

- First determine the inplane $\theta = 0$ frequency spectrum.

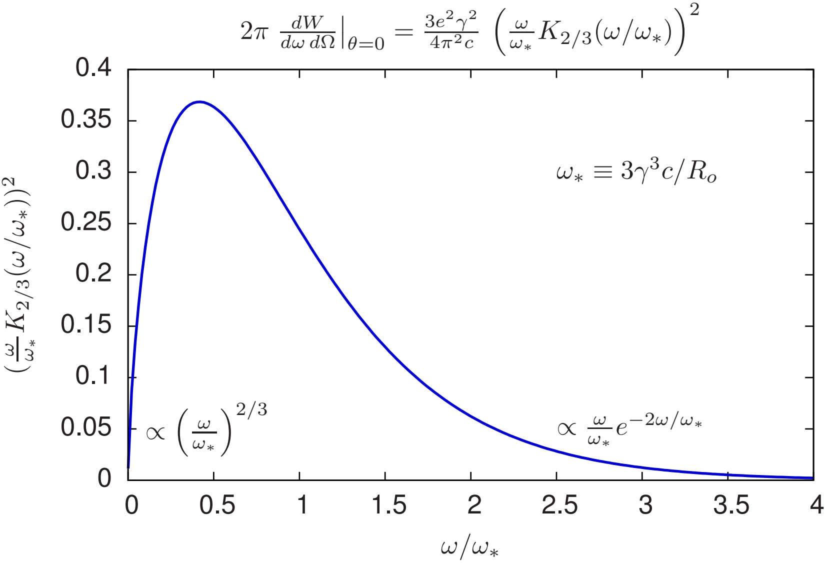
Then, $$\xi = \frac{\omega R_0}{3c}\left(\frac{1}{\gamma^2} + \theta^2\right)^{3/2} \xrightarrow[\theta\to 0]{} \frac{\omega R_0}{3c\gamma^3} \equiv x \text{ , so}$$

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{3q^2}{4\pi^2 c}\gamma^2 \left(x K_{2/3}(x)\right)^2$$

So $2\pi\, dW/d\omega d\Omega \big|_{\theta=0}$ in units $\frac{3e^2\gamma^2}{4\pi^2 c}$



$x \equiv \frac{1}{3}\frac{\omega R}{c\gamma^3}$ = frequency in units


$2\pi \left.\frac{dW}{d\omega\, d\Omega}\right|_{\theta=0} = \frac{3e^2\gamma^2}{4\pi^2 c} \left(\frac{\omega}{\omega_*} K_{2/3}(\omega/\omega_*)\right)^2$
$\omega_* \equiv 3\gamma^3 c/R_o$
$\propto \left(\frac{\omega}{\omega_*}\right)^{2/3}$
$\propto \frac{\omega}{\omega_*} e^{-2\omega/\omega_*}$
$(\frac{\omega}{\omega_*} K_{2/3}(\omega/\omega_*))^2$
ω/ω_*
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.5
1
1.5
2
2.5
3
3.5
4