Last Time

Discussed Radiation During Collisions

For grazing collisions $\beta_2 \approx \beta_1 + \delta \beta$, and $\phi$ is often random.

Found that

$$\vec{E}(\omega) = \frac{1}{4\pi r^2} e^{ikr} \left( \frac{n \times n \times \hat{V}_2}{(1 - n \cdot \beta_2)} - \frac{n \times n \times \hat{V}_1}{(1 - n \cdot \beta_1)} \right)$$

More generally decompose the outgoing light into two polarization vectors

$$\vec{E} = \vec{E}_\parallel \hat{\epsilon}_\parallel + \vec{E}_\perp \hat{\epsilon}_\perp$$

Then more generally $\epsilon$'s can be complex, for example:

$$\epsilon = (0, 1, i, 0)$$ records circularly light
Last time Continued

The properties that we derived from the analysis of waves are,
\[ \vec{E} = E_1 \vec{e}_1 + E_2 \vec{e}_2 \]

\[ \vec{e}_a \cdot \vec{e}_b = \delta_{ab} \quad \text{orthogonal} \]

\[ \hat{n} \cdot \vec{E} = 0 \quad \text{transverse to direction} \]

Then we write the energy per frequency per solid angle with polarization

\[ \frac{2\pi}{\omega d\Omega} = c \left| \vec{E} \right|^2 r^2 \]

\( \parallel \) (in \( \hat{n}, \hat{\beta} \) plane) \hspace{1cm} \text{and} \hspace{1cm} \perp \text{ (out of} \hat{n}, \hat{\beta} \text{ plane)} \]

\[ \frac{2\pi}{\omega d\Omega} = c \left| \vec{E}_1 \right|^2 r^2 \]

Then since for this example \( \hat{n} \times \hat{n} \times \hat{\beta} \) is already transverse we have

\[ \vec{E} = -\frac{q}{4\pi\epsilon_0 c^2} \left[ \vec{E} \times \hat{\beta} - \vec{E} \cdot \hat{\beta} \right] \]

And the frequency spectrum is

\[ \frac{2\pi}{\omega d\Omega d\Omega} = \frac{q}{16\pi\epsilon_0^2} \left| \frac{\vec{E} \cdot \hat{\beta}}{(1 - \alpha \cdot \beta_2)} - \frac{\vec{E} \cdot \hat{\beta}_1}{(1 - \alpha \cdot \beta_1)} \right|^2 \]

You will need this for homework.
Scattering of Radiation

\[ E_{\text{escatt}} \]

\[ E_{\text{inc}} \]

\[ ? \text{ radiated fields by induced current} \]

Incoming Light \( \lambda \) small Sphere

\[ \rightarrow \]

\[ a \]

\[ \rightarrow \] take small

- Initially study the scattering of radiation by small objects \( \lambda \gg a \)

- Why does light scatter? The light induces currents in the object. The induced currents radiate.

So, if you want to know how much light was scattered, you should first compute how the incoming field accelerates the charge, and then compute how the accelerated charges radiate.

- For an extended object, the acceleration in one point can create fields at another which influences the other points leading to a complicated standing wave.

Treat small objects
Suggests Approximations

1) Small objects. \( \lambda \gg a \). The field can be considered constant over the extent of object.

2) Weak scattering. The induced fields, \( \mathbf{E}_{\text{scatt}} \), are small compared to \( \mathbf{E}_{\text{inc}} \).
Cross Sections

\[ E_{\text{scatt}} = C \omega e^{i(k \cdot r - \omega t)} \]

\[ E = E_0 e^{i(k \cdot r - \omega t)} \]

The constant is a vector which depends on direction and is \( \propto E_0 \).

Then

\[ E = E_{\text{inc}}(r) + E_{\text{scatt}}(r) \]

"Scattering amplitude"

The incoming energy is:

\[ \overline{\mathcal{S} \cdot E} = \frac{1}{2} c \left| E_{\text{inc}} \right|^2 \] — time averaged energy per area per time

The outgoing energy per solid angle

\[ \frac{dP(\mathbf{E})}{d\Omega} = r^2 \frac{\mathbf{S} \cdot n^2}{\mathbf{E}} = \frac{1}{2} c |F|^2 \mathbf{E}_{\text{scatt}}^* \cdot \mathbf{E}_{\text{scatt}} \]

So define the cross section

\[ \frac{d\sigma}{d\Omega} (\mathbf{E} \cdot \mathbf{E}_0) = \frac{1}{2} c |F|^2 \mathbf{E}_{\text{scatt}}^* \cdot \mathbf{E}_{\text{scatt}} = \left| \mathbf{E} \cdot F(k) \right|^2 \propto (\text{meters}^2) \]

energy scattered (per solid angle + polarization \( \mathbf{E} \)) per incoming flux.
Thomson Cross Section

Light electron scattering

\[ \sigma = \langle \frac{\text{Power radiated}}{\frac{1}{2} |E_{\text{inc}}|^2} \rangle = \frac{P}{\frac{1}{2} |E_{\text{inc}}|^2} \]

For an electron experiencing acceleration \(a\)

\[ P = \frac{q^2 2}{4 \pi} \frac{a^2}{3 c^3} \]

The force/accel is determined by the incoming light

\[ a = q \frac{E_{\text{inc}}}{m} e^{iwt} \]

Then

\[ a^2 = \frac{1}{2} |a|^2 = \frac{q^2}{m^2} \frac{1}{2} |E_{\text{inc}}|^2 \]
Thus

\[
\sigma = \frac{q^2}{4\pi} \frac{2}{3c^3} \frac{q^2}{m^2} \frac{1}{2} |E_{\text{inc}}|^2
\]

= \frac{8\pi}{3} \left( \frac{q^2}{4\pi mc^2} \right)^2 = \frac{8\pi}{3} r_e^2

This is known as the classical electron radius

\[ r_e = \frac{q^2}{4\pi mc^2} \]

Note

\[ r_e = \left( \frac{q^2}{4\pi \hbar c} \right) \left( \frac{\hbar}{mc} \right) = \lambda_c \text{ (by } 2\pi \text{)} \]

\[ \lambda_c = \frac{\hbar}{mc} \]

Numerically

\[ \lambda_c = \frac{\hbar c}{m_ec^2} = \frac{19\text{eV nm}}{0.5\text{ MeV}} \]

Then

\[ r_e = 2.8\text{ fm} = 2.8 \text{ femtometers} = 2.8 \text{ fermi} \]
The cross section is

\[ \sigma = \frac{8 \pi \gamma_e^2}{3} = 66 \text{ fm}^2 \]

\[ = 660 \text{ milli barn} \]

\[ = 0.66 \text{ barns} \quad \text{units of cross section} \]

**Polarization**

\[ \epsilon_\parallel \quad n \]

\[ \epsilon_\perp \]

\[ \theta \]

Want to show that the polarized cross section is

\[ \frac{d\sigma}{d\Omega} (\epsilon; \epsilon_0) = \gamma_e^2 |\epsilon^* \cdot \epsilon_0|^2 \]
\[ E_{\text{rad}} = n \times n \times \frac{1}{c} \oint \frac{dA_{\text{rad}}}{c} = \frac{q}{\pi} n \times n \times \frac{\alpha(t_e)}{4\pi r c^2} \]

Let's rederive this result, by approximating Lienard-Wiechert

\[ A_{\text{rad}} = \frac{q}{4\pi r} \frac{V(t)}{c} \left(1 - n \cdot V(t)/c\right) \]
\[ T = t - \frac{r}{c} + \frac{n \cdot r_e(t)}{c} \approx t - \frac{r}{c} = t_e \]

The non-rel approximation replaces \( T \approx t_e = t - \frac{r}{c} \) and expands \( V/c \ll 1 \),

\[ A_{\text{rad}} \approx \frac{q}{4\pi r} \frac{V(t_e)}{c} \]

And so

\[ E_{\text{rad}} = \frac{q}{\pi} n \times n \times \alpha(t_e) \frac{1}{4\pi r c^2} \]

The acceleration is along \( \hat{\varepsilon}_0 \)

\[ \hat{a} = \hat{\varepsilon}_0 \frac{q}{m} E_0 e^{-i\omega t} \]

Then we want to compute

\[ |E^* E_{\text{rad}}|^2 \propto |E^* (n \times n \times \hat{a})|^2 \]

\( \uparrow \) final polarization
Using $\mathbf{b}(ac) - (ab)\mathbf{c}$

$$\mathbf{E}^\ast \cdot (\mathbf{n} \times \mathbf{n} \times \mathbf{a}) = \mathbf{E}^\ast \cdot (-\hat{a} + \frac{n}{n \cdot \hat{a}} (n \cdot \hat{a}))$$

$$= -\mathbf{E}^\ast \cdot \hat{a} \quad \text{(since } \mathbf{E}^\ast \text{ is transverse)}$$

To $\mathbf{n}$ it projects out the longitudinal pieces.

Then

$$\left| \mathbf{E} \cdot \mathbf{n} \times \mathbf{n} \times \mathbf{a} \right|^2 = \frac{q^2}{\frac{1}{4}c^2 m^2} \frac{2}{2} \left| \mathbf{E}_0 \right|^2 \left| \mathbf{E}^\ast \cdot \mathbf{E}_0 \right|^2$$

So we compute:

$$\frac{d\sigma (\mathbf{E}^\ast, \mathbf{E}_0)}{d\Omega} = \frac{dP/d\Omega}{\frac{c}{2} |\mathbf{E}_0|^2} = \frac{\frac{1}{2} c^2 |\mathbf{E}^\ast \cdot \mathbf{E}_\text{rad}|^2}{\frac{1}{2} c |\mathbf{E}_0|^2}$$

$$= \left( \frac{\frac{q^2}{4\pi mc^2}}{2} \right)^2 \left| \mathbf{E}^\ast \cdot \mathbf{E}_0 \right|^2$$

$$\equiv r_e^2$$

$$= r_e^2 |\mathbf{E}^\ast \cdot \mathbf{E}_0|^2$$
Transverse and parallel polarized cross section

There are four cases here:

1. $\varepsilon^\| \cdot \varepsilon^\| = \cos \theta$ (see figure)

2. $\varepsilon^\perp \cdot \varepsilon^\perp = 1$

3. $\varepsilon^\| \cdot \varepsilon^\perp = 0$

4. $\varepsilon^\perp \cdot \varepsilon^\perp = 0$

So the cross section for initially unpolarized light (i.e. $\varepsilon^0$ is 50% of time $\| + 50\%$ of time $\perp$) to produce light polarized in the $\|$ or $\perp$ direction is:

$$\frac{d\sigma^\|}{d\Omega} = \frac{1}{2} \left[ \frac{d\sigma}{d\Omega} (\varepsilon^\|, \varepsilon^\|) + \frac{d\sigma}{d\Omega} (\varepsilon^\|, \varepsilon^\perp) \right]$$

$$= \frac{1}{2} r^2 \cos^2 \theta$$

$$\frac{d\sigma^\perp}{d\Omega} = \frac{1}{2} \left[ \frac{d\sigma}{d\Omega} (\varepsilon^\perp, \varepsilon^\|) + \frac{1}{2} \frac{d\sigma}{d\Omega} (\varepsilon^\perp, \varepsilon^\perp) \right]$$

$$= \frac{1}{2} r^2$$
Transverse and $\parallel$ polarized cross sections pg. 2

The cross section to produce light of any polarization by initially unpolarized light is

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega} + \frac{d\sigma}{d\Omega}$$

$$= \Gamma \left( \frac{1 + \cos^2\Theta}{2} \right)$$

The degree of polarization depends on the angle

$$\text{degree of pol} = \frac{d\sigma - d\sigma}{d\sigma + d\sigma} = \frac{(1 - \cos^2\Theta)}{(1 + \cos^2\Theta)}$$

1

| 0° | 90° | 180° |

degree of pol.

Question: Why is the light 100% transversely polarized at 90°?

Ans.: At 90° the current is up and down for the parallel case. Thus there is no component of the current transverse
Ans: continued...

to the observation. Thus the cross section
for this case vanishes.

\[
\begin{align*}
\theta &= 90^\circ \\
\uparrow \hspace{4cm} \uparrow \\
\text{up and down acceleration}
\end{align*}
\]