Last Time

$$\nabla \cdot E = \rho$$

$$\nabla \times B = \frac{J}{c} + \frac{1}{c}\partial_t E$$

$$\nabla \cdot B = 0$$

$$-\nabla \times E = \frac{1}{c}\partial_t \vec{B}$$

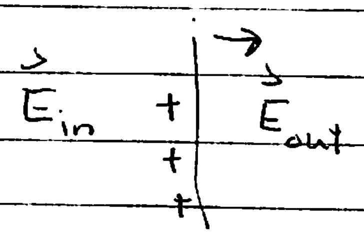
Taking $c \to \infty$ we found, electrostatics:

$$\left.\begin{array}{l} \nabla \cdot E = \rho \\ \nabla \times E = 0 \end{array}\right\} \quad \text{and} \quad \vec{B} = 0, \quad \text{or} \quad \begin{array}{l} -\nabla^2 \varphi = \rho \\ E = -\nabla\varphi \end{array}$$

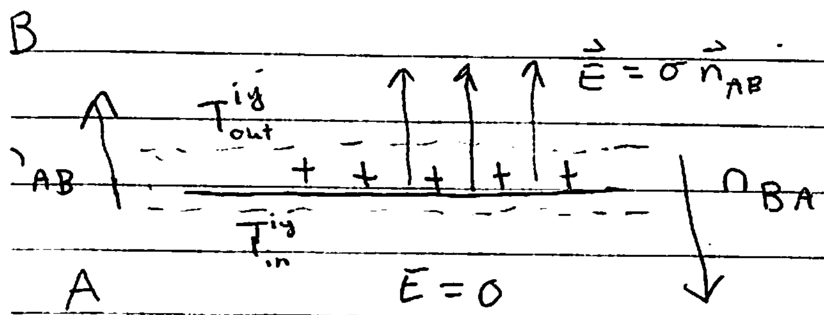
Discussed the boundary conditions of electro-statics

$$n \cdot (\vec{E}_{out} - \vec{E}_{in}) = \sigma$$

$$\vec{n} \times (\vec{E}_{out} - \vec{E}_{in}) = 0$$

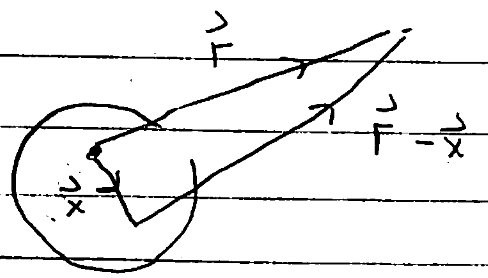


Then discussed the forces on metal



$$\frac{\text{net force}}{\text{area}} = \underbrace{\cancel{(n_{AB})_i T^{iy}_{in}}}_{\substack{\text{force}\\\text{from inner}\\\text{surf}\\\text{A on B}}} \overset{0}{} + \underbrace{(n_{BA})_i T^{iy}_{out}}_{\substack{\text{force from}\\\text{out}\\\text{B on A}}} = \frac{\sigma^2}{2}(n_{AB})^j$$

Multipole Expansion



find potential for $r \gg x$

$$\star \quad \varphi(\vec{r}) = \int \frac{\rho(\vec{x})\, d^3\vec{x}}{4\pi|\vec{r}-\vec{x}|}$$

$$(1+u)^\alpha = 1 + \alpha u + \frac{\alpha(\alpha-1)}{2!}u^2 + \dots$$

$$\frac{1}{|\vec{r}-\vec{x}|} = \frac{1}{(r^2+x^2-2\vec{r}\cdot\vec{x})^{-1/2}} = \frac{1}{r}\left(1 - \frac{2\vec{r}\cdot\vec{x}}{r^2} + \frac{x^2}{r^2}\right)^{-1/2}$$

$$\frac{1}{|\vec{r}-\vec{x}|} \simeq \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{x}}{r^2} + \left(\frac{3}{2}\frac{(\vec{r}\cdot\vec{x})(\vec{r}\cdot\vec{x})}{r^4} - \frac{1}{2}\frac{x^2}{r^2}\right) + \dots\right)$$

$$\simeq \frac{1}{r} + \frac{\hat{r}^i x_i}{r^2} + \frac{1}{2}\frac{\hat{r}^i\hat{r}^j}{r^3}\left(3x_ix_j - \delta_{ij}x^2\right) + \dots$$

Now take this expression and subsitute into $\star$

$$\varphi(r) = \frac{1}{4\pi}\left[\frac{Q_{TOT}}{r} + \frac{\hat{r}^i p_i}{r^2} + \frac{1}{2}\frac{\hat{r}^i\hat{r}^j}{r^3}Q_{ij} + O\left(\frac{1}{r^4}\right)\right]$$

So

$$Q_{TOT} = \int d^3x \, \rho(x) \longleftarrow$$ monopole moment
scalar

$$(\vec{p})^i = \int d^3x \, \rho(x) \, x^i \longleftarrow$$ dipole moment
vector

$$Q_{ij} = \int d^3x \, \rho(x) \, (3x_i x_j - \delta_{ij} x^2)$$

Symmetric traceless
quadrupole tensor

Fields

$$\vec{E}_{mono} = \frac{Q_{TOT}}{4\pi r^2} \hat{r} \quad \propto \quad \frac{1}{r^2}$$

$$\vec{E}_{dipole} = \frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{4\pi r^3} \propto \frac{1}{r^3}$$

$$\vec{E}_{Quad} \sim \text{dont know} \quad \propto \quad \frac{1}{r^4}$$

How to solve the Poisson Equation

Want to solve:

$$-\nabla^2 \varphi(x) = \rho(x)$$

First limit ourselves to free space $\Phi \xrightarrow[x\to\infty]{} 0$
Then a modest amount of intuition says

$$\varphi(\vec{r}) = \int \frac{\rho(\vec{r}_0)}{4\pi|\vec{r}-\vec{r}_0|}$$

Thats right. Formally compute a green fcn:

$$-\nabla_r^2 G_0(r, r_0) = \delta^3(\vec{r}-\vec{r}_0)$$

(Aside:

infinitely narrow spike in 3D

$$\int d^3r\, \delta^3(\vec{r}-\vec{r}_0)\, f(\vec{r}) = f(r_0) \qquad \int d^3r\, \delta^3(\vec{r}-\vec{r}_0) = 1$$

)

Then solution is the convolution of the Green fcn

$$\varphi(r) = \int d^3r_0\, G_0(\vec{r}, \vec{r}_0)\, \rho(\vec{r}_0)$$

and the charge density

Since

$$-\nabla^2\varphi = \int d^3x_0 \underbrace{-\nabla_x^2\, G(x,x_0)}_{\delta^3(x-x_0)}\, \rho(x_0)$$

$$-\nabla^2\varphi = \rho(x)$$

<u>More physically,</u> the Green-fcn $G(x,x_0)$ is the potential at x due to a unit point charge at x_0

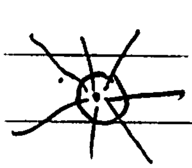
For free space we know the answer:

$$G(\vec{x},\vec{x}_0) = \frac{1}{4\pi|\vec{x}-\vec{x}_0|}$$

Verify that:

$$-\nabla^2 \frac{1}{4\pi r} = 0 \quad \text{except at } r=0$$

and check:



$$\int_{\substack{\text{vol small}\\ \text{ball}}} \vec{\nabla}\cdot\overbrace{\left(-\vec{\nabla}\frac{1}{4\pi r}\right)}^{E=\frac{\hat{r}}{4\pi r^2}} d^3r = \int_{\text{area}} \vec{E}\cdot\overbrace{\hat{r}\, r^2 d\Omega}^{d\vec{S}_{\text{ball}}}$$

$$= 1$$

So

$$-\nabla^2 \frac{1}{4\pi r} = \delta^3(\vec{r})$$

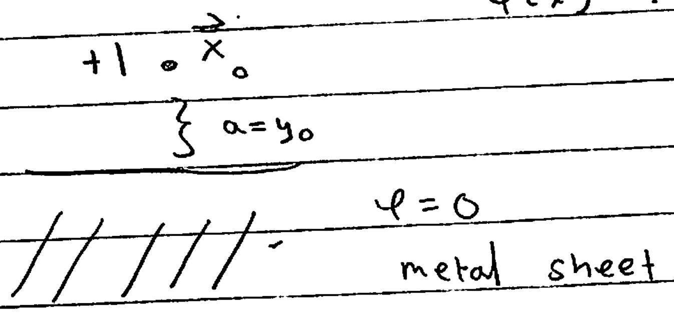
Now

$$\varphi(x) = \int d^3x_0 \; G(x, x_0) \, \rho(x_0)$$

$$\varphi(x) = \underbrace{\int d^3x_0 \; \frac{\rho(x_0)}{4\pi |\vec{x} - \vec{x}_0|}}_{\text{perhaps clear!}}$$

olving for Green-fcn (Images)

$\varphi(\vec{x}) = ?$

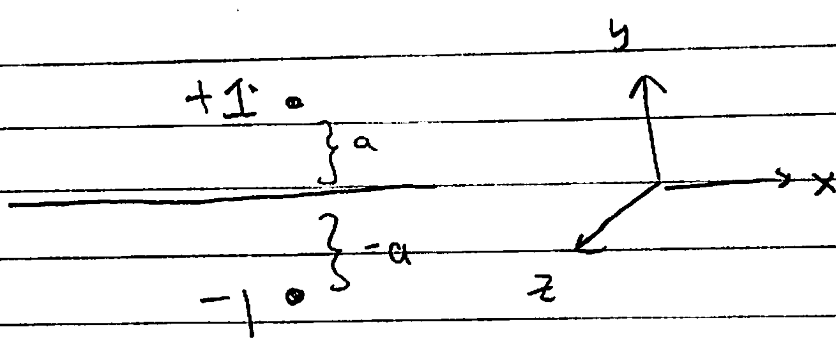


- Want to solve for $G(\vec{x}, \vec{x}_0)$

$$-\nabla^2 G(\vec{x}, x_0) = \delta^3(\vec{x} - \vec{x}_0)$$

- Together with BC $\varphi = 0$ at $z = 0$

Solution - place an "image" charge at $y_I = -a$ with opposite sign



This is the self field

The potential $\overbrace{G_0 \text{ free green fcn}}$

$$G(\vec{x}, \vec{x}') = \frac{1}{4\pi|\vec{x}-x'|} + \overbrace{\frac{1}{4\pi|\vec{x}-\vec{x}'_I|}}^{\text{regular in upper region}}$$

$\vec{x}' = (x', y', z')$ $\quad \vec{x}'_I = (x', y', z')$

only this φ_{in} part is responsible for the force on charge

Note that one always finds:

$$\boxed{G(x, \vec{x}') = G_0(x, \vec{x}') + \varphi_{ind}(\vec{x})}$$

Where $-\nabla^2 G = -\nabla^2 G_0(x, x') = \delta^3(\vec{x}-x')$, implying

that $-\nabla^2 \varphi_{ind}(x) = 0$. obeys the homogeneous equation.

U_{int} = interaction energy between plane and charge q at x'

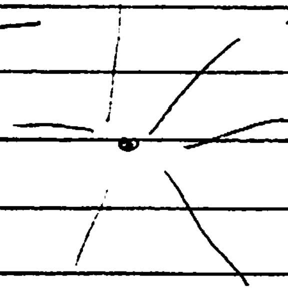
$= q\,\varphi_{ind}(x')$

$$\boxed{U_{int} = \lim_{x \to x'} q\left[G(x, x') - G_0(x, x')\right]}$$

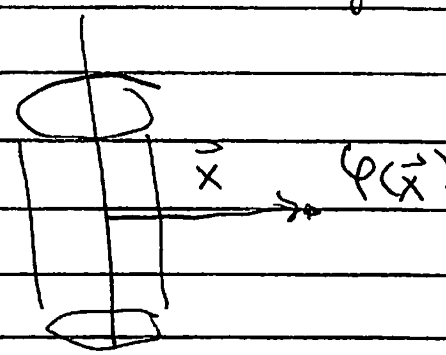
Similarly Force is:

$$\vec{F} = q\,\vec{E}_{ind}(x') = q \lim_{x \to x'} \left[-\nabla_x G(x, x') - (-\nabla_x G_0(x, x'))\right]$$

In 2D Free Space



line of charge



Use gauss law to show; $\vec{x}' = 0$

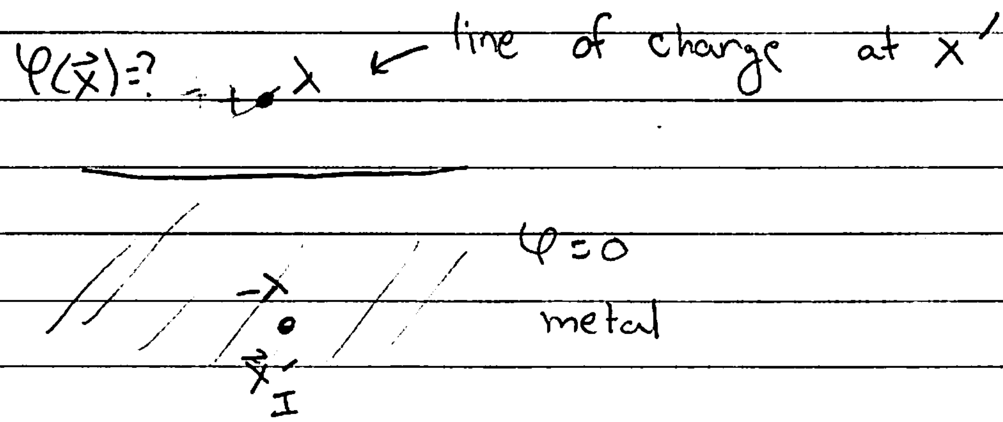
$$\varphi(\vec{x}) = -\frac{\lambda}{2\pi} \log(|\vec{x}|) + \text{Const}$$

Since the green fcn is the potential at $\vec{x}$ due to a point charge at x'

$$G(x, x') = -\frac{1}{2\pi} \log |\vec{x} - \vec{x}'|$$

$$\varphi(\vec{x}) = -\int d^2\vec{x}' \, \rho(x') \frac{-1}{2\pi} \log |\vec{x} - \vec{x}'|$$

In 2D



$$\varphi(x) = G(\vec{x}, \vec{x}') = \frac{-1}{2\pi} \log |\vec{x} - \vec{x}'| + \frac{1}{2\pi} \log |\vec{x} - \vec{x}'_I|$$

potential at $\vec{x}$ due to a point charge at $\vec{x}'$