Last Time

\[ \nabla \cdot E = \rho \]

\[ \nabla \times B = \frac{\dot{\rho}}{c} + \frac{\dot{E}}{c} \]

\[ \nabla \cdot B = 0 \]

\[ -\nabla \times E = \frac{\dot{\rho}}{c} \frac{\dot{B}}{c} \]

Taking \( c \to \infty \) we found electrostatics:

\[ \nabla \cdot E = \rho \]
\[ \nabla \times E = 0 \]
\[ \nabla \times B = 0 \]

\[ \text{and } \dot{B} = 0, \text{ or } -\nabla^2 \psi = \rho \]

\[ \psi \]

Discussed the boundary conditions of electrostatics

\[ n \cdot (E_{\text{out}} - E_{\text{in}}) = \sigma \]

\[ \nabla \times (\frac{\partial}{\partial t} E_{\text{out}} - \frac{\partial}{\partial t} E_{\text{in}}) = 0 \]

Then discussed the forces on metal

\[ \vec{B} \]

\[ \vec{T}_{\text{out}} \]

\[ \vec{T}_{\text{in}} \]

\[ \sigma \vec{n}_{AB} \]

\[ A \]

\[ \vec{E} = 0 \]
\[
\text{net force} = \frac{(n_{\text{in}}) \cdot T_{\text{in}} + (n_{\text{out}}) \cdot T_{\text{out}}}{\text{area}} = \frac{\sigma^2 (N_{AB})^3}{2}.
\]
Multipole Expansion

\[ \varphi(\vec{r}) = \int \frac{\rho(\vec{x})}{4\pi |\vec{r} - \vec{x}|} \, d^3x \]

\[ (1 + u)^n = 1 + nu + \frac{n(n-1)}{2!} u^2 + \ldots \]

\[ \frac{1}{|\vec{r} - \vec{x}|} = \frac{1}{(r^2 + x^2 - 2 \vec{r} \cdot \vec{x})^{1/2}} = \frac{1}{r} \left( 1 - \frac{2 \vec{r} \cdot \vec{x}}{r^2} + \frac{x^2}{r^2} \right)^{1/2} \]

\[ \frac{1}{|\vec{r} - \vec{x}|} \cdot \frac{1}{r} = \frac{1}{r} \left( 1 + \frac{\vec{r} \cdot \vec{x}}{r^2} + \left( \frac{3}{2} \frac{(\vec{r} \cdot \vec{x})(\vec{r} \cdot \vec{x})}{r^4} - \frac{1}{2} \frac{x^2}{r^2} \right) + \ldots \right) \]

\[ \alpha = \frac{1}{r} + \frac{\hat{r} \cdot \vec{x}}{r^2} + \frac{1}{2} \frac{\vec{r} \cdot \vec{r}}{r^3} \left( 3 \vec{x} \vec{r} - \delta_{ij} x^2 \right) + \ldots \]

Now take this expression and substitute into \( \varphi \)

\[ \varphi(r) = \frac{1}{4\pi} \left[ \frac{Q_{\text{tot}}}{r} + \frac{\hat{r} \cdot \vec{p}_i}{r^2} + \frac{1}{2} \frac{\hat{r} \cdot \hat{r} \cdot \hat{r}}{r^3} Q_{ij} + O\left( \frac{1}{r^4} \right) \right] \]
So

\[ Q_{\text{tot}} = \int d^3x \rho(x) \leftarrow \text{monopole moment scalar} \]

\[ (\vec{p})^i = \int d^3x \rho(x) x^i \leftarrow \text{dipole moment vector} \]

\[ Q_{ij} = \int d^3x \rho(x) (3x_i x_j - \delta_{ij} x^2) \]

\[ \text{Symmetric traceless quadrupole tensor} \]

Fields

\[ \vec{E} = \frac{Q_{\text{tot}}}{4\pi r^2} \hat{r} \propto \frac{1}{r^2} \]

\[ \vec{E}_{\text{dipole}} = \frac{3(\vec{p} \cdot \hat{r}) \hat{r} - \hat{p}}{4\pi r^3} \propto \frac{1}{r^3} \]

\[ \vec{E}_{\text{quad}} \sim \text{don't know} \propto \frac{1}{r^4} \]
How to solve the Poisson Equation

Want to solve:

\[-\nabla^2 \psi(x) = \rho(x)\]

First limit ourselves to free space \( x \to \infty \)
Then a modest amount of intuition says

\[
\psi(r) = \int \frac{\rho(r_0)}{4\pi |r - r_0|} \, dr_0
\]

That's right. Formally compute a green function:

\[-\nabla_r^2 G_0(r, r_0) = \delta^3(r - r_0)\]

(Aside:

\[
\int d^3 r \, \delta^3(r - r_0) \, f(r) = f(r_0) \quad \int d^3 r \, \delta^3(r - r_0) = 1
\]

Then solution is the convolution of the green function

\[
\psi(r) = \int d^3 r_0 \, G_0(r, r_0) \, \rho(r_0)
\]

and the charge density
Since
\[- \nabla^2 \varphi = \int \frac{d^3 x}{8^3(x-x_0)} \nabla^2 G(x, x_0) \rho(x_0) \]

\[- \nabla^2 \varphi = \rho(x) \quad G(x, x_0) \]

More physically, the Green-fcn is the potential at \( x \) due to a unit point charge at \( x_0 \).

For free space we know the answer:

\[ G(x, x_0) = \frac{1}{4\pi |x-x_0|} \]

Verify that:

\[ - \frac{1}{4\pi r} = 0 \quad \text{except at } r = 0 \]

and check:

\[ E = \frac{\hat{r}}{4\pi r^2} \int \frac{d^3 s}{8^3 \text{ball}} \]

\[ \int \vec{\nabla} \cdot \left( - \frac{\vec{\nabla} \frac{1}{4\pi r} \right) - d^3 r = \int \vec{E} \cdot \hat{r} r^2 d\Omega \]

\[ \text{Vol small ball} \quad \text{area} \]

\[ = 1 \]
So

$$-\nabla^2 \frac{1}{r} = \delta^3(\hat{r})$$

Now

$$\Phi(x) = \int d^3x_0\, G(x, x_0)\, \rho(x_0)$$

$$\Phi(x) = \int d^3x_0\, \frac{\rho(x_0)}{4\pi |x - x_0|}$$

perhaps clear?
Solving for Green's fn (Images)

\[ \Phi (\vec{x}) = ? \]

\[ +1 \cdot \vec{x}_0 \]

\[ \sum \alpha = y_0 \]

// // // \ - \ y = 0 \metal sheet

- Want to solve for \( G(x, x_0) \)

\[-\nabla^2 G(x, x_0) = \delta^3(x-x_0) \]

- Together with BC \( \Phi = 0 \) at \( z = 0 \)

Solution - place an "image" charge at \( y_1 = -a \) with opposite sign

\[ +1 \star \]

\[ \sum y_0 \]

\[ -1 \star \]

\[ z = -a \]

\[ x \]

\[ y \]
The potential $G_0$, free green function regular in upper region:

$$G(x, x') = \frac{1}{4\pi|\hat{x} - \hat{x}'|} + \frac{1}{4\pi|\hat{x} - \hat{x}'|}$$

$\hat{x}' = (x', y', z')$, $\hat{x} = (x, y, z)$

Only this part is responsible for the force on charge.

Note that one always finds:

$$G(x, x') = G_0(x, x') + \Phi_{\text{ind}}(x')$$

Where $-\nabla^2 G = -\nabla^2 G_0(x, x') = \delta^3(\hat{x} - \hat{x}')$, implying that $-\nabla^2 \Phi_{\text{ind}}(x) = 0$. Obey the homogeneous equation.

$$U_{\text{int}} = \text{interaction energy between plane and charge } q \text{ at } x'$$

$$U_{\text{int}} = \lim_{x \to x'} q \left[ G(x, x') - G_0(x, x') \right]$$

Similarly, force is:

$$F = q E_{\text{ind}}(x') = q \lim_{x \to x'} \left[ -\nabla G(x, x') - (-\nabla_x G_0(x, x')) \right]$$
In 2D Free Space line of charge

\[ \vec{x} = (x, y) \]

Use Gauss's law to show: \( \vec{x} = a \)

\[ \varphi(\vec{x}) = -\frac{\lambda}{2\pi} \log \left( 1^{\vec{x}'} \right) + \text{Const} \]

Since the Green function is the potential at \( \vec{x} \) due to a point charge at \( \vec{x}' \)

\[ G(x, x') = -\frac{1}{2\pi} \log |x - x'| \]

\[ \varphi(\vec{x}) = -\int d^2x' \rho(\vec{x}') -\frac{1}{2\pi} \log |\vec{x} - \vec{x}'| \]
In 2D

\[ \phi(x) = \pm \chi \quad \text{line of charge at } x' \]

\[ \phi = 0 \quad \text{metal} \]

\[ \phi(x) = G(x, x') = \frac{-1}{2\pi} \log |x - x'| + \frac{1}{2\pi} \log |x - x'| \]

potential at \( x \) due to a point charge at \( x' \)