Consider a charged shell of charge per solid angle $S(\theta, \phi)$. Determine the potential everywhere.

$$\Phi(r) = ? \quad \psi(r) = ?$$

$$\rho(\vec{r}) = \frac{1}{4\pi} \frac{S(r - r_o)}{r^2} S(\theta, \phi)$$

1. We will expand $S(\theta, \phi)$ in spherical harmonics

$$S(\theta, \phi) = \sum_{lm} S_{lm} Y_{lm}(\theta, \phi)$$

2. In the limit that:

$$S(\theta, \phi) = \delta(\cos \theta - \cos \theta_o) S(\phi - \phi_o)$$

The $\rho(\vec{r}) = \delta^3(\vec{r} - \vec{r}_o)$ and $S_{lm} = Y_{lm}^*(\theta_o, \phi_o)$, i.e.

$$S(\theta, \phi) = \sum_{lm} Y_{lm}^*(\theta_o, \phi_o) Y_{lm}(\theta, \phi)$$

$$\psi(r) = \frac{1}{4\pi |\vec{r} - \vec{r}_o|}$$
Problem

- Given a charged sphere of radius $R_0$ with charge per solid angle $S(\theta, \phi)$ determine the potential everywhere.

Plan:

- Separate variables, solve inside and outside, integrate across the shell to match the inside and outside.

Solution:

Inside, hence $r < R$:

$$-\nabla^2 \psi = 0$$

Notice that if $\psi = R(r) Y(\theta, \phi)$

$\uparrow$ Coords $\uparrow$

$t$ to surf $t$ to surface
Then we compute:

\[- \frac{1}{\varphi} \nabla^2 \varphi \quad \text{with} \quad \nabla^2 = - \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \]

And find

\[- \frac{1}{2} \frac{2}{r^2 \partial R}{\partial r} + \frac{1}{2} L^2 Y = 0 \quad \text{if } r \text{ is fixed} \]

Thus we are led to consider the eigenvalue equation

\[L^2 Y = \lambda Y \quad \lambda_n = l(l+1)\]

We know this eigenvalue problem, the operator is hermitian and the eigenfns are complete and orthonormal.

Thus at each \(r\) we can expand the solution

\[\varphi(r) = \sum_{\ell m} R_{\ell m}(r) Y_{\ell m}(\theta, \varphi)\]

And adjust the \(R_{\ell m}(r)\) to match the solution across the shell.
Then from \(-\nabla^2 \psi = 0\)

\[
\left( -\frac{1}{2} \frac{\partial^2}{\partial r^2} + \frac{L^2}{r^2} \right) \sum R_{lm} Y_{lm} = 0
\]

Leads to

\[
\left[ -\frac{1}{2} \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} \right) + \frac{L(L+1)}{r^2} \right] R_{lm}^{in}(r) = 0
\]

The solution to this equation are \(r = 0, r = \infty\) regular singular point

\[
R_{lm}^{in}(r) = A_{lm} r^l + B_{lm} \frac{1}{r^{l+1}}
\]

Similarly, the solution outside the shell are written

\[
\psi_{\text{out}} = \sum_{lm} R_{lm}^{\text{out}}(r) Y_{lm}(\theta, \phi)
\]

\[
R_{lm}^{\text{out}}(r) = A_{lm} r^l + B_{lm} \frac{1}{r^{l+1}}
\]
Now:

for $r \to 0$ want a regular solution

$$B_{lm}^{\text{in}} = 0$$

for $r \to \infty$ want a regular solution

$$A_{lm}^{\text{out}} = 0$$

So for the remaining two conditions

$$A_{lm}^{\text{in}}, B_{lm}^{\text{out}}$$

we demand continuity of $\phi$, and require that in each surface element

$$\vec{n} \cdot \vec{E}_{\text{out}} - \vec{n} \cdot \vec{E}_{\text{in}} = 0$$

This is derived by integrating the Poisson equation from $R - \epsilon$ to $R + \epsilon$. 
\[ p = \frac{S(\theta, \phi)}{r^2} S(r-r_0) \]

So that

\[ \oint d^3r \rho(r) = \oint d^2r S(\theta, \phi) \]

Then the Poisson Equation

\[ \left[ -\frac{1}{r^2} \frac{\partial}{\partial r} \frac{1}{r^2} \frac{\partial}{\partial r} + \frac{l(l+1)}{r^2} \right] \sum_{l} \sum_{m} \frac{R_l^m(r) Y_l^m(\theta, \phi)}{r^2} = \frac{S(\theta, \phi)}{r^2} S(r-r_0) \]

So expanding \( S(\theta, \phi) \) in the same basis

\[ S(\theta, \phi) = \sum_{l} \sum_{m} S_{l}^{m} Y_{l}^{m}(\theta, \phi) \]

So

\[ \left[ -\frac{1}{r^2} \frac{\partial}{\partial r} \frac{1}{r^2} \frac{\partial}{\partial r} + \frac{l(l+1)}{r^2} \right] R_{l}^{m}(r) = \frac{S_{l}^{m}}{r^2} S(r-r_0) \]

Now multiply by \( r^2 \) and integrate from \( r = r_0 - \varepsilon \)

to \( r = r_0 + \varepsilon \)

\[ \int_{r_0-\varepsilon}^{r_0+\varepsilon} 2 \frac{r^2}{r^2} \frac{\partial}{\partial r} R_{l}^{m} = \int_{r_0-\varepsilon}^{r_0+\varepsilon} 2 \frac{r^2}{r^2} \frac{\partial}{\partial r} R_{l}^{m} \]

\( l(l+1) \) term gives \( O(\varepsilon) \) since \( R \) is continuous
So find

\[ \star - \left( \frac{d}{dr} \left( \frac{R_{\text{out}}^{\ell m}}{\ell m} - r^2 \frac{\partial R_{\text{in}}^{\ell m}}{\partial r} \right) \right) = S^{\ell m}_{\text{jump}} \text{ (jump condition)} \quad r = r_0 \]

This is equivalent to \( n \cdot E_{\text{out}} - n \cdot E_{\text{in}} = \sigma \)

So with continuity and jump equation

\[ B_{\ell m}^{\text{out}} = A_{\ell m}^{\text{in}} \frac{r^\ell}{r_0^{\ell+1}} \quad \text{(continuity)} \]

\[ (l+1) \frac{B_{\ell m}^{\text{out}}}{r_0^l} + l A_{\ell m}^{\text{in}} \frac{r_0^{l+1}}{r_0^l} = S_{\ell m}^{\text{jump}} \]

Find

\[ B_{\ell m} = \frac{S_{\ell m}}{2l+1} r_0^l \]

\[ A_{\ell m} = \frac{S_{\ell m}}{2l+1} \frac{1}{r_0^{l+1}} \]

So

This is it. It expresses \( \psi \) in terms of \( S(\theta, \phi) \)

\[ \psi(r) = \sum_{\ell m} \frac{S_{\ell m}}{2l+1} \left( \frac{r_0}{r} \right)^{l+\frac{1}{2}} Y_{\ell m}^{\dagger} (\theta, \phi) \quad r > r_0 \]

\[ \psi(r) = \sum_{\ell m} \frac{S_{\ell m}}{2l+1} \left( \frac{r}{r_0} \right)^{l+\frac{1}{2}} Y_{\ell m}^{\dagger} (\theta, \phi) \quad r < r_0 \]
This gives the potential for any source specified by

\[ S(\theta, \phi) = \sum_{\ell m} S_{\ell m} Y_{\ell m}(\theta, \phi) \]

For \( s_{\ell m} = Y_{\ell m}^*(\theta, \phi_0) \), this a point charge (see overview).

\[ \Phi(r) = \frac{1}{4\pi} = \sum_{\ell m} \frac{r_0^{\ell}}{r^{(\ell + 1)/2}} \frac{1}{\Gamma} Y_{\ell m}(\theta, \phi_0) Y_{\ell m}(\theta, \phi) \quad r > r_0 \]

### Important Points

1. Identify coords \( r \) (i.e. \( r \)) and parallel \((\theta, \phi)\) to surface where b.c. are specified.

2. Solve eigenvalue eqn for parallel directions. These are complete & orthogonal.

3. Expand solution in these eigen-fcns and solve for \( r \) direction. General homogeneous solution

\[ \Phi = \sum_{\ell m} \left( A_{\ell m} r^\ell + B_{\ell m} \right) \frac{Y_{\ell m}(\theta, \phi)}{\Gamma_{\ell + 1}} \]

4. Adjust coefficients so boundary conditions are satisfied. Integrate across \( \ell m \)s with second order eqs to determine jump conditions.
For example

$$\left[ -\frac{1}{2} \frac{r^2 \partial^2}{r^2 \partial r} + \frac{l(l+1)}{r^2} \right] g_{\ell \ell}(r, r_0) = \frac{1}{r^2} \delta(r-r_0)$$

- find $g$ for $r < r_0$ and $g$ for $r > r_0$

- Jump $-r^2 \frac{\partial^2 g_{\text{out}}}{\partial r^2} + r^2 \frac{\partial^2 g_{\text{in}}}{\partial r^2} = f \cdot + \text{continuity}$

$$g_{\ell \ell}(r, r_0) = \frac{1}{2l+1} \left[ \frac{g_{\ell \ell}^R \Theta(r-r_0) + g_{\ell \ell}^L \Theta(r_0-r)}{r_0} \right]$$

$r > r_0$

$r < r_0$