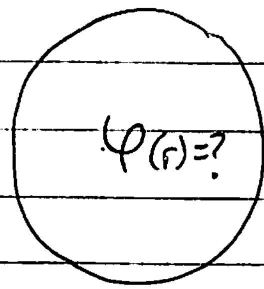
Overview Charged Shell

Consider a charged shell of charge per solid angle $S(\theta,\phi)$. Determine the potential everywhere.



$\varphi(r) = ?$

$$\rho(\vec{r}) = \frac{1}{r^2}\,\delta(r - r_0)\, S(\theta,\phi)$$

① We will expand $S(\theta,\phi)$ in spherical harmonics

$$S(\theta,\phi) = \sum_{lm} S_{lm}\, Y_{lm}(\theta,\phi)$$

② In the limit that:

$$S(\theta,\phi) = \delta(\cos\theta - \cos\theta_0)\,\delta(\phi - \phi_0)$$

The $\rho(\vec{r}) = \delta^3(\vec{r} - \vec{r}_0)$ and $S_{lm} = Y^*_{lm}(\theta_0, \phi_0)$, i.e.

$$S(\theta,\phi) = \sum_{lm} Y^*_{lm}(\theta_0,\phi_0)\, Y_{lm}(\theta,\phi)$$

$$= \delta(\cos\theta - \cos\theta_0)\,\delta(\phi - \phi_0)$$

And $\varphi(r) = \dfrac{1}{4\pi|\vec{r} - \vec{r}_0|}$

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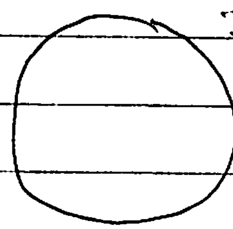
Problem

- Given a charged sphere^shell of radius r_0 with charge per solid angle $\underset{\sim}{S}(\theta,\phi)$ determine the potential everywhere



$\varphi = ?$

Plan:

- Separate variables, solve inside & outside, integrate across the shell to match the inside and outside.

Solution:

Inside have, $r < R$:

$$-\nabla^2 \varphi = 0$$

Notice that if $\varphi = R(r)\,Y(\theta, \varphi)$

$R(r)$: coord $\perp$ to surf

$Y(\theta, \varphi)$: coords $\parallel$ to surface

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Then we compute:

angles only

$$-\frac{r^2}{\varphi}\nabla^2\varphi \quad \text{with} \quad -\nabla^2 = -\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}L^2$$

And find

$$\underbrace{-\frac{1}{R}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r}}_{\text{if } r \text{ is fixed}} + \underbrace{\frac{-1}{Y}L^2 Y}_{\text{Then this is constant}} = 0 \qquad L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Thus we are led to consider the eigenvalue eqauation

$$L^2 Y_n = \lambda_n Y_n \qquad \lambda_n = \ell(\ell+1)$$

We know this eigenvalule problem, the operator is hermitian and the eigenfcns are complete & orthogom.

Thus at each r we can expand the solution

$$\varphi(r) = \sum_{\ell m} R_{\ell m}(r)\, Y_{\ell m}(\theta, \varphi)$$

And adjust the $R_{\ell m}(r)$ to match the solution accross the shell

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Then from $-\nabla^2 \varphi_{in} = 0$

$$\left(-\frac{1}{r}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} L^2 \right) \sum R_{\ell m} Y_{\ell m} = 0$$

Leads to

$$\left[-\frac{1}{r}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} \ell(\ell+1) \right] R^{in}_{\ell m}(r) = 0$$

The solution to this equation are

two regular singul point $r=0$ $r=\infty$

$$R^{in}_{\ell m}(r) = A^{in}_{\ell m} r^{\ell} + \frac{B^{in}_{\ell m}}{r^{\ell+1}}$$

Similarly the solution outside the shell are written

$$\varphi^{out} = \sum_{\ell m} R^{out}_{\ell m}(r) Y_{\ell m}(\theta,\phi)$$

with

$$R^{out}_{\ell m}(r) = A^{out}_{\ell m} r^{\ell} + \frac{B^{out}_{\ell m}}{r^{\ell+1}}$$

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Now:

for $r \to 0$ want a regular solution

$$B^{in}_{lm} = 0$$

for $r \to \infty$ want a regular solution

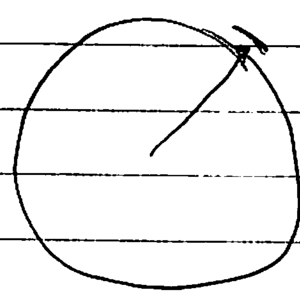
$$A^{out}_{lm} = 0$$

So for the remaining two conditions

A^{in}_{lm} B^{out}_{lm}.

we demand continuity of φ, and relquire that in each surface element

$$\underbrace{\vec{n} \cdot E_{out} - \vec{n} \cdot E_{in} = \sigma}$$



this is derived by integrating the poisson equation from $R - \varepsilon$ to $R + \varepsilon$

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$$\rho = \frac{S(\theta,\phi)}{r^2}\,\delta(r-r_0)$$

So that

$$\int d^3\vec{r}\,\rho(\vec{r}) = \int d\Omega\, S(\theta,\phi)$$

Then the Poisson Equation

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}L^2\right]\sum_{lm} R_{lm}(r)\,Y_{lm}(\theta,\phi) = \frac{S(\theta,\varphi)}{r^2}\,\delta(r-r_0)$$

So expanding $S(\theta,\phi)$ in the same basis

$$S(\theta,\phi) = \sum_{lm} S_{lm} Y_{lm}(\theta,\varphi)$$

So

this a prototype eq for 1D green fcns

$$\boxed{\left[-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{l(l+1)}{r^2}\right] R_{lm}(r) = \frac{S_{lm}}{r^2}\,\delta(r-r_0)}$$

Now multiply by r^2 and integrate from $r = r_0 - \varepsilon$ to $r = r_0 + \varepsilon$

- $$\int_{r_0-\varepsilon}^{r_0+\varepsilon} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} R_{lm} = r^2 \frac{\partial R^{out}_{lm}}{\partial r} - r^2 \frac{\partial R^{in}_{lm}}{\partial r}$$

- $l(l+1)$ term gives $O(\varepsilon)$ since R_{lm} is continuous

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So find

$$\star \qquad -\left(r^2 \frac{\partial R^{out}_{\ell m}}{\partial r} - r^2 \frac{\partial R^{in}_{\ell m}}{\partial r} \right)\Bigg|_{r=r_0} = S_{\ell m} \qquad \text{(jump condition)}$$

This is equivalent to $n \cdot E^{out} - n \cdot E^{in} = \sigma$
So with continuity and jump equation

$$\frac{B^{out}_{\ell m}}{r_0^{\ell+1}} = A^{in}_{\ell m} r^{\ell} \qquad \text{(continuity)}$$

$$(\ell+1) \frac{B^{out}_{\ell m}}{r_0^{\ell}} + \ell A^{in}_{\ell m} r_0^{\ell+1} = S_{\ell m} \qquad \text{(jump)}$$

Find

$$B_{\ell m} = \frac{S_{\ell m}}{2\ell+1} r_0^{\ell}$$

$$A_{\ell m} = \frac{S_{\ell m}}{2\ell+1} \frac{1}{r_0^{\ell+1}}$$

So This is it. It expresses φ in terms of $S(\theta,\phi)$

$$\boxed{\begin{aligned} \varphi(\vec{r}) &= \sum_{\ell m} \frac{S_{\ell m}}{2\ell+1} \left(\frac{r_0}{r}\right)^{\ell} \frac{1}{r} Y_{\ell m}(\theta,\phi) \qquad r > r_0 \\ \varphi(\vec{r}) &= \sum_{\ell m} \frac{S_{\ell m}}{2\ell+1} \left(\frac{r}{r_0}\right)^{\ell} \frac{1}{r_0} Y_{\ell m}(\theta,\phi) \qquad r < r_0 \end{aligned}}$$

This gives the potential for any source specified by

$$S(\theta, \varphi) = \sum_{lm} S_{lm} Y_{lm}(\theta, \phi)$$

For $S_{lm} = Y^*_{lm}(\theta_0, \phi_0)$, this a point charge (see overview)

$$\varphi(r) = \frac{1}{4\pi|\vec{r} - \vec{r}_0|} = \sum_{lm} \left(\frac{r_0}{r}\right)^l \frac{1}{r} Y_{lm}(\theta_0, \phi_0) Y_{lm}(\theta, \phi) \qquad r > r_0$$

Important Points

1. Identify coords $\perp$ (i.e. r) and parallel (θ, φ) to surface where b.c. are specified

2. Solve eigenvalue eqn for parallel directions these are complete & orthogonal

3. Expand solution in these eigen-fcns and solve for $\perp$ direction

general homogeneous solution

$$\boxed{\varphi = \sum_{lm} \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right) Y_{lm}(\theta, \phi)}$$

4. Adjust coefficients so boundary conditions are satisfied. Integrate across δ fcns with second order eqs to determine jump conditions.

For example

$$\boxed{-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{r^2}} g_\ell(r, r_0) = \frac{1}{r^2}\delta(r-r_0)$$

- find g for $r<r_0$ and g for $r>r_0$
- Jump $-r^2\frac{\partial}{\partial r}g^{out} + r^2 \frac{\partial}{\partial r} g^{in} = 1$ + continuity

$$g_\ell(r, r_0) = \frac{1}{2\ell+1}\left[\underbrace{\frac{r_0^\ell}{r^{\ell+1}}\Theta(r-r_0)}_{r>r_0} + \underbrace{\frac{r^\ell}{r_0^{\ell+1}}\Theta(r_0-r)}_{r<r_0}\right]$$