Today

- Forces on multipoles
- Three examples: one easy, two medium

Last Time

- Differential equation (a wide class of 2nd order Diffeq)

$$\left[-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] y(x) = 0$$

$p(x)$ positive

- $p(x)\, W(x) = \text{const}$

$$p(x)\left[y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x) \right] = \text{Const}$$

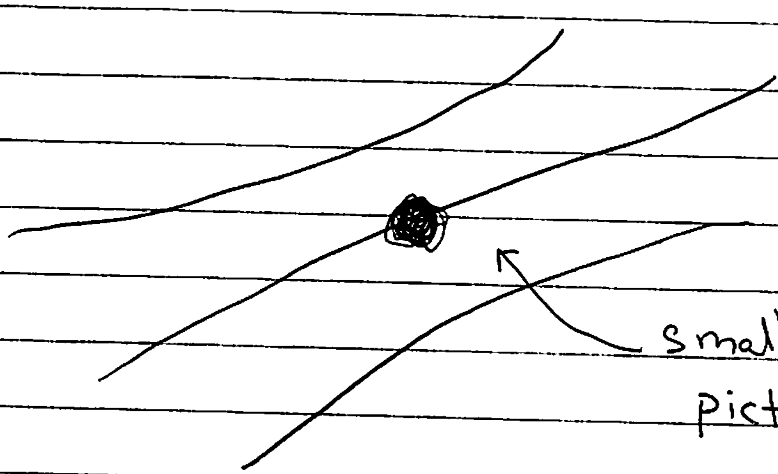
- Constructed Green Fcn

$$\left[-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] G(x, x_0) = \delta(x - x_0)$$

$$G(x, x_0) = C\, y_{out}(x_>)\, y_{in}(x_<)$$

$$= \frac{y_{out}(x_>)\, y_{in}(x_<)}{p(x_0)\, W(x_0)}$$

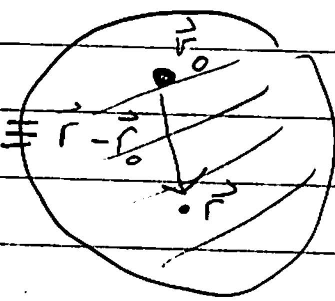
Forces on Multipoles



Long wavelength external field $\vec{E}(\vec{r}_0)$, or Φ_{ext}, $-\nabla^2 \Phi_{ext} = 0$

small object. Magnify this picture:

$\delta\vec{r} \equiv \vec{r} - \vec{r}_0$



$$U_{int} = \int \rho(\vec{r})\, \Phi_{ext}(\vec{r})\, d^3r$$

Write

$$\Phi(\vec{r}) = \Phi(r_0) + \frac{\partial \Phi(r_0)}{\partial r_0^i}\,\delta r^i + \frac{1}{2}\frac{\partial \Phi(r_0)}{\partial r^i \partial r^j}\overbrace{\delta r^i \delta r^j}^{\text{reducable}} + \ldots$$

Use:

$$-\frac{\partial}{\partial r^i}\frac{\partial}{\partial r_i}\Phi = 0 \qquad \text{(Laplace equation)}$$

or $$\delta^{ij}\frac{\partial}{\partial r^i}\frac{\partial}{\partial r^j}\Phi = 0$$

So write:

$$\Phi(\vec{r}) = \Phi(\vec{r}_0) + \frac{\partial \Phi}{\partial r^i}\delta r^i + \frac{1}{2}\frac{\partial^2 \Phi(r_0)}{\partial r_0^i \partial r_0^j}\left(\delta r^i \delta r^j - \frac{1}{3}\delta^{ij}\delta r^2\right)$$

So Find

$$U_{int} = \Phi(r_0) Q_{TOT} + \frac{\partial \Phi}{\partial r^i} p^i + \frac{1}{6}\frac{\partial \Phi}{\partial r^i \partial r^j} Q^{ij} + \dots$$

Where

$$Q_{TOT} = \int_r \rho(\vec{r})$$

$$(\vec{p}) = \int_r \rho(\vec{r})\, \delta \vec{r}$$

$$Q^{ij} = \int_r \rho(r)\left(3\,\delta r^i \delta r^j - \delta^{ij}\delta r^2\right)$$

Or

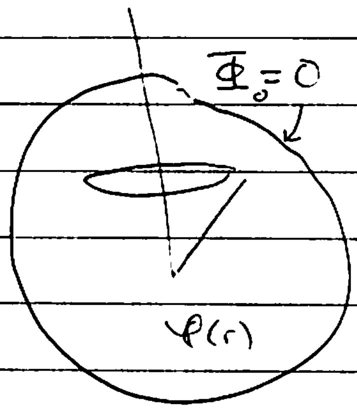
$$\boxed{U_{int}^{(r_0)} = \Phi(r_0) Q_{TOT} - \vec{p} \cdot \vec{E}(\vec{r}_0) - \frac{1}{6} Q^{ij}\, \partial_i E_j (r_0)}$$

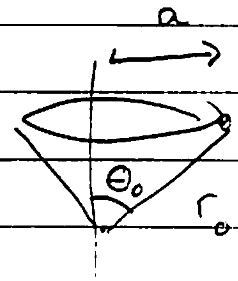
To find the force:

$$\vec{F} = -\nabla U_{int}$$

You will need this for homework

A worked example pg. 1





A ring of radius, a, and constant charge per length λ, sits inside a grounded metal sphere of radius R. Determine the force on the ring.

The charge density is

$$\rho = \frac{\lambda a}{r^2} \delta(r - r_0)\, \delta(\underbrace{\cos\theta}_{\equiv x} - \underbrace{\cos\theta_0}_{\equiv x_0})$$

So

$$Q = \int r^2 \overset{dr}{\vee} d(\cos\theta)\, d\phi \;\; \rho(r)$$

$$Q = \int r^2 dr\, dx\, d\phi \;\; \frac{\lambda a}{r^2} \delta(r - r_0)\, \delta(x - x_0)$$

$$Q = \lambda\, 2\pi a \quad \checkmark$$

A worked example pg. 2

So

$$-\nabla^2 \varphi = \lambda a \frac{1}{r^2} \delta(r-r_0)\,\delta(\cos\theta - \cos\theta_0)$$

φ is, up to a constant λa the green fcn in the space of fcns which are azimuthally symmetric. Define

$$\varphi = \lambda a\, G(\vec{r}).$$

Where $$-\nabla^2 G = \frac{1}{r^2} \delta(r-r_0)\,\delta(\cos\theta - \cos\theta_0) \qquad (\text{Eq.}\,\star)$$

- want to expand in eigen fcns for the coordinates $\parallel$ to where the B.C. are applied and solve for $\perp$ direction. Recall:

① $$-\nabla^2 = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{L^2}{r^2}$$

② $$L^2 P_\ell(\cos\theta) = \ell(\ell+1) P_\ell(\cos\theta)$$

$$P_\ell \propto Y_{\ell 0}$$

③ Completeness

$$\sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{2} P_\ell(\cos\theta)\, P_\ell(\cos\theta_0) = \delta(\cos\theta - \cos\theta_0)$$

A worked expample pg. 3

So try, $$G = \sum_{\ell} g_\ell(r)\, \overbrace{P_\ell(\cos\theta)}^{x}\, \overbrace{P_\ell(\cos\theta_0)}^{x_0}\, \frac{2\ell+1}{2}$$

Find plugging into ☆ on previous page

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{r^2}\right] g_\ell(r) = \frac{1}{r^2}\,\delta(r-r_0)$$

So we need to solve for $r < r_0$ and $r > r_0$

can choose norm constant

$$y_{in} = A r^\ell + \cancel{\frac{B}{r^{\ell+1}}} \Rightarrow y_{in} = \left(\frac{r}{R}\right)^\ell$$

For $r > r_0$. Need that at $r = R$ potential is zero

$$y_{out} = A r^\ell + \frac{B}{r^{\ell+1}}$$

$$y_{out} = -\left(\frac{r}{R}\right)^\ell + \left(\frac{R}{r}\right)^{\ell+1}$$

Then

$$p(r)\,W(r) = r^2\left(y_o y_{in}' - y_{in} y_o'\right) = -R(2\ell+1)$$

So $$g(r, r_0) = \frac{1}{R(2\ell+1)}\left(\frac{r_<}{R}\right)^\ell \left(\left(\frac{R}{r_>}\right)^{\ell+1} - \left(\frac{r_>}{R}\right)^\ell\right)$$

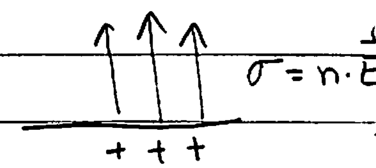
$$\boxed{\varphi(r, r_0) = \sum_\ell \frac{\lambda a}{2R} P_\ell(x) P_\ell(x_0) \left[\left(\frac{r_<}{R}\right)^\ell \left(\left(\frac{R}{r_>}\right)^{\ell+1} - \left(\frac{r_>}{R}\right)^\ell\right)\right]}$$

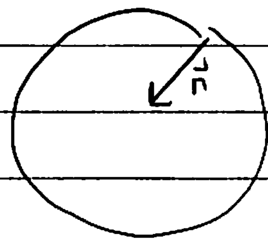
A. worked example pg. 4

- The total induced charge on the surface of the sphere should be:

$$-Q = -\lambda\,(2\pi a)$$

- The induced charge





$$\sigma = \vec{n}\cdot\vec{E} = -\hat{r}\cdot(-\nabla\varphi)$$

$$= +\left.\frac{\partial\varphi}{\partial r}\right|_{r=R}$$

Now with $r_> = r$ and $r_< = r_0$ find; doit

$$\sigma = \left.\frac{\partial\varphi}{\partial r}\right|_{r=R} = \frac{-Q}{4\pi R^2}\sum_\ell P_\ell(x)P_\ell(x_0)\,(2\ell+1)\left(\frac{r_0}{R}\right)^\ell \qquad \left(\text{Eq } ☆\right)$$

used $\lambda a = Q/2\pi$

So integrating

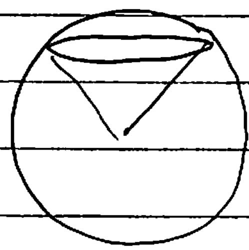
$$Q_{ind} = \int R^2\,d\Omega\,\sigma = R^2\int_{-1}^{1}\underset{\uparrow\ \cancel{dx = \sin\theta\, d\theta}}{dx}\int_0^{2\pi} d\phi\,\sigma$$

$$= 2\pi R^2\int_{-1}^{1}dx\left[\frac{-Q}{4\pi R^2}\sum_\ell P_\ell(x)P_\ell(x_0)(2\ell+1)\left(\frac{r_0}{R}\right)^\ell\right.$$

$$Q_{ind} = -Q$$

Only $\ell = 0$ contributes to integral

Also check that as $r_0 \to R$



Then the induced charge should approach

charge per area of a ring

$$\sigma = -\frac{Q}{2\pi R^2}\,\delta(\cos\theta - \cos\theta_0)$$

Setting $r_0 = R$ in Eq. ☆ on previous page:

$$\sigma\Big|_{r_0=R} = \frac{-Q}{4\pi R^2}\,\underbrace{\sum_\ell P_\ell(x)\,P_\ell(x_0)\,(2\ell+1)\cdot 1}_{=2\,\delta(\cos\theta - \cos\theta_0)}$$

completeness

$$\sigma\Big|_{r_0=R} = \frac{-Q}{2\pi R^2}\,\delta(\cos\theta - \cos\theta_0)$$

The interaction energy:

$$U_{int} = \frac{1}{2}\int_{\vec{r}} \rho(\vec{r})\,\Phi_{ind}(\vec{r})$$

Where

$$\Phi_{ind} = \Phi(\vec{r}) - \Phi_0(\vec{r})$$

potential with sphere

potential w. out sphere Just a ring of charge

$$U_0 = \frac{1}{2}\int_{\vec{r}} \rho(r)\,\Phi_0(r)$$

is the energy required to assemble the ring in free space.

You can compute Φ_0 in exactly the same way, except

$$y_{out} = \left(\frac{R}{r}\right)^{\ell+1} \quad \text{instead of} \quad y_{out} = \left(\frac{R}{r}\right)^{\ell+1} - \left(\frac{r}{R}\right)^{\ell}$$

So

$$\Phi_0 = \sum_\ell \frac{\lambda a}{2R} P_\ell(x) P_\ell(x_0) \left(\frac{r_<}{R}\right)^{\ell}\left(\frac{R}{r_>}\right)^{\ell+1}$$

and

$$\Phi_{ind} = \Phi - \Phi_0 = \sum_\ell \frac{\lambda a}{2R} P_\ell(x) P_\ell(x_0) \left(\frac{r_<}{R}\right)^{\ell}\left(\frac{r_>}{R}\right)^{\ell}$$

$$= \sum_\ell -\frac{\lambda a}{2R} P_\ell(x) P_\ell(x_0) \left(\frac{r\,r_0}{R^2}\right)^{\ell} \quad \leftarrow \text{regular}$$

So U_{int} is, using $\lambda a = Q/2\pi$

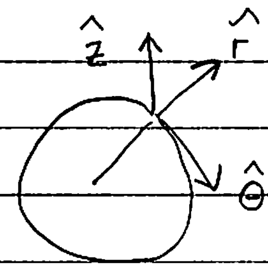
$$U_{int} = \frac{1}{2}\int r^2 \, d(\cos\theta)\, d\phi \left[\frac{Q}{2\pi} \frac{1}{r^2} \delta(r-r_0)\, \delta(\cos\theta - \cos\theta_0) \right] \times \Phi_{ind}(r,\theta)$$

$$U_{int} = \frac{1}{2} Q \, \Phi_{ind}(r_0, \theta_0)$$

$$\boxed{U_{int} = -\frac{Q^2}{8\pi R} \sum_{\ell} \left(P_\ell(x_0)\right)^2 \left(\frac{r_0}{R}\right)^{2\ell}}$$

The Force:

$$F^z = \hat{z} \cdot (-\nabla U_{int})\Big|_{\text{on ring}}$$



$$\hat{z} = \cos\theta\, \hat{r} - \sin\theta\, \hat{\theta}$$

$$-\nabla U = -\hat{r}\frac{\partial U}{\partial r} + \hat{\theta}\,\frac{-1}{r}\frac{\partial U}{\partial \theta}$$

And $\quad -\hat{z}\cdot\nabla U = -\cos\theta\, \partial U/\partial r + \dfrac{\sin\theta}{r_0}\dfrac{\partial U}{\partial \theta} = F^z$

Using

$$-\cos\theta \frac{\partial U}{\partial r} = \frac{Q^2}{8\pi R}\sum_l 2l\left(\frac{r_0}{R}\right)^{2l-1}\frac{1}{R}\,\overbrace{\cos\theta\, P_l^2(\cos\theta_0)}^{x\, P_l^2}$$

$$\frac{\sin\theta}{r_0}\frac{\partial U}{\partial \theta} = \frac{Q^2}{8\pi R^2}\sum_l\left(\frac{r_0}{R}\right)^{2l-1}\frac{1}{R}\,\underbrace{2\sin^2\theta\; P_l'(\cos\theta_0)\, P_l(\cos\theta_0)}_{\cancel{(1-x^2)P_l' P_l}}$$

Using recurrence relation:

$$(1-x^2)P_l'(x) = l\,P_{l-1}(x)\;\overbrace{-\;l x\, P_l(x)}^{\text{cancels with } -\cos\theta\frac{\partial U}{\partial r}}$$

Find

$$\boxed{F^z = \frac{Q^2}{4\pi R^2}\sum_{l=1}^{\infty} l\, P_l(x_0)\, P_{l-1}(x_0)\left(\frac{r_0}{R}\right)^{2l-1}}$$

ring radius

$a = \frac{1}{2} R$ Sphere Radius

Force