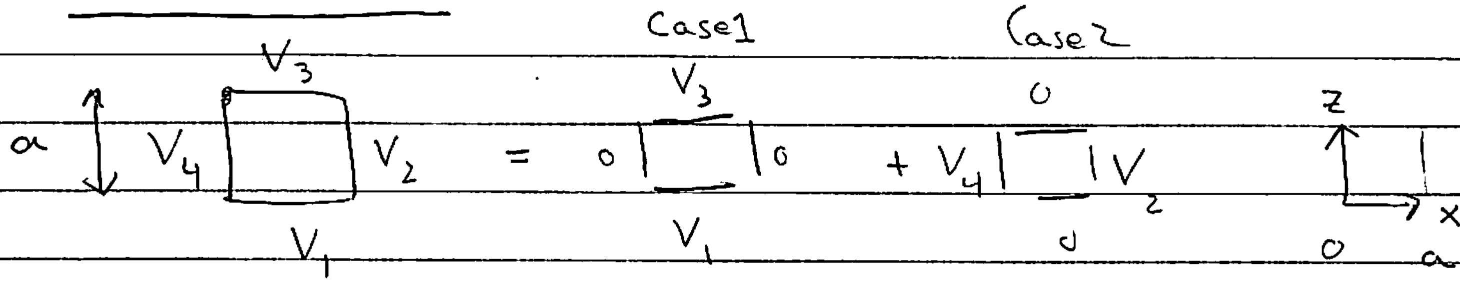
Cartesian Coords (2D)



What is the potential of a square, all sides held at a different potential. Write as a super-position. Treat case 1.

Try separation $X(x)Z(z) = \varphi(x,z)$. Calculate

$$-\frac{1}{\varphi}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = 0$$

- $$\underbrace{-\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{if } x \text{ is held fixed}} + \underbrace{-\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\text{This is const}} = 0$$

- $$\frac{d^2X}{d^2x^2} = -k^2X \qquad \frac{d^2Z}{dz^2} = +k^2 Z$$

- At this point k^2 is arbitrary. But, we are imposing two boundary conditions at ends and norm is arbitrary

$$X(0) = X(a) = 0$$

only for certain values of k

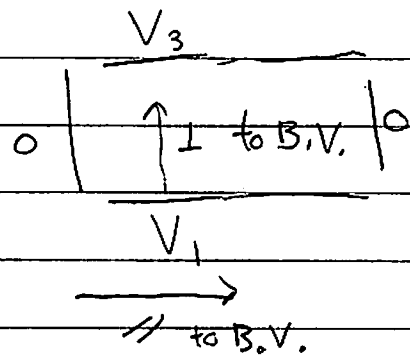
Cartesian Cords

will X satisfy the separated eqs & bc

$$X = \cancel{A \cos kx} + B \sin kx$$

$$\psi_n(x) \equiv X_n = B \sin k_n x \qquad k_n = \frac{n\pi}{a} \qquad n = 1, 2,$$

Notice structure:



Integrating // equations we are specifying the two endpoints. X must fit into the box. Thats an eigenvalue eqn, fixing k_n and X_n. The eigen functions $\psi_n(x)$ are complete (for functions satisfying b.c

For given k_n solve the $\perp$ eqs for Z:

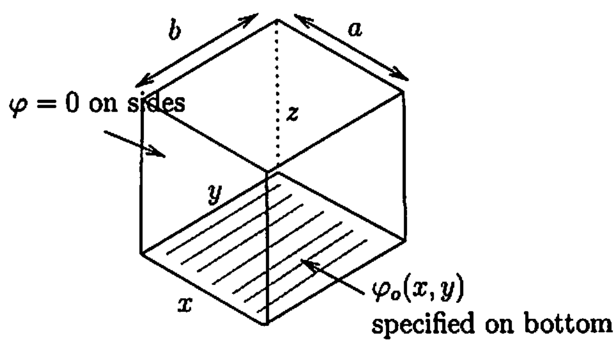
$$Z_n = A e^{-k_n z} + B e^{+k_n z}$$

$$\Phi = \sum_n \left(A_n e^{-k_n z} + B_n e^{k_n z} \right) \overbrace{\psi_n(x)}^{\sin(k_n x)} \quad \leftarrow \text{complete.}$$

adjust A_n and B_n to get boundary values V_1, V_3 at $z=0$ and $z=a$

D Separation of Variables

D.1 Cartesian coordinates



(a) Laplacian

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi = 0 \tag{D.1}$$

(b) Eigen fucntions along boundary vanishing at $x = 0$ and $x = a$ and $y = 0$ and $y = b$

$$\psi_{nm}(x, y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \qquad n = 1\ldots\infty \qquad m = 1\ldots\infty$$

(c) Orthogonality

$$\int_0^a dx \int_0^b dy\, \psi_{nm}\, \psi_{n'm'} = \left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\delta_{nn'}\delta_{mm'}$$

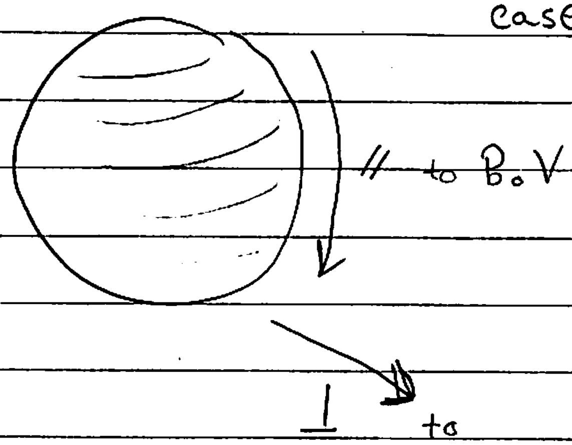
(d) Solution

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{nm}e^{-\gamma_{nm}z} + B_{nm}e^{+\gamma_{nm}z}\right]\psi_{nm}(x, y) \tag{D.2}$$

where $\gamma_{nm} = \sqrt{(n\pi/a)^2 + (m\pi/b)^2}$

Spherical Coords

Treat the azimuthally symmetric case.



$$-\nabla^2 = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right]$$

Or using $x \equiv \cos\theta$ $\quad \sin\theta\frac{\partial}{\partial\theta} = (1-x^2)\frac{\partial}{\partial x}$

$$-\nabla^2 = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\left[-\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}\right] = 0$$

Try separation $\varphi = R(r)\mathcal{P}(x)$. Evaluate $\frac{r^2}{\varphi}\nabla^2\varphi = 0$, gives throug separation procedure

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{r^2}\right] R(r) = 0$$

$$\star \quad \left[-\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}\right]\mathcal{P}(x) = -\ell(\ell+1)\mathcal{P}(x)$$

So far ℓ is completely arbitrary. We are however applying to B.C. at the north and south pole, that function should be regular, i.e, that $P(x)$ should fit on surface of sphere. This constrains the allowed values of ℓ to be integer. We will not describe it in detail here. But this is how it works:

it is therefore a hypergeometric function

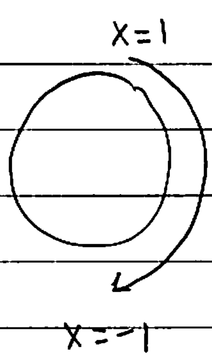
① The differential equation* has three regular-singular points $(+1, -1, \infty)$. Near each singular point, take $+1$ for example, there will be one regular solution and one irregular solution. The general solution

$$\mathcal{P} = A\, P_\ell(x) + B\, Q_\ell(x)$$

legendre function — regular at $x=1$ (irregular at $x=\infty$)

associated legendre function — irregular at $+1$ (regular at ∞)

② Integrating from the top of the to the bottom. The solution which is regular at the top will become a mixture of the regular and irregular solution at the bottom, i.e. it wont fit on the sphere, except for certain values of ℓ



③ Only for certain values of l will the solution be regular at $x=-1$. In the current case the series solution for $P_l(x)$, near $x=1$

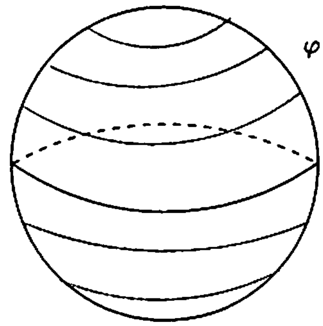
$$P_\nu(x) = 1 + \frac{-\nu(1+\nu)}{2}(z-1) + \frac{(-\nu)(1-\nu)(1+\nu)(2+\nu)}{16}(z-1)^2$$

will terminate for ν = integer. This is typical of hypergeometric series for certain values.

So find that, the eigen fcns are $P_l(x)$ and the general solution is

$$\Phi = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) \underbrace{P_l(\cos\theta)}_{\psi_l(\cos\theta)}$$

D.2 Spherical coordinates



(a) Laplacian

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\Phi = 0 \tag{D.3}$$

(b) Eigen fucntions along boundary θ, ϕ, regular at $\theta = 0$ and π, 2π periodic in ϕ

$$\psi_{\ell m}(\theta, \phi) = Y_{\ell m}(\theta, \phi) \qquad \ell = 0 \ldots \infty \quad m = -\ell \ldots \ell$$

(c) Orthogonality:

$$\int d\Omega\, Y^*_{\ell m}(\theta, \phi)\, Y_{\ell' m'}(\theta, \phi) = \delta_{\ell\ell'}\delta_{mm'}$$

(d) Solution

$$\Phi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[A_{\ell m}r^\ell + \frac{B_{\ell m}}{r^{\ell+1}}\right]Y_{\ell m} \tag{D.4}$$

(e) When there is no azimuthal dependence things simplify to

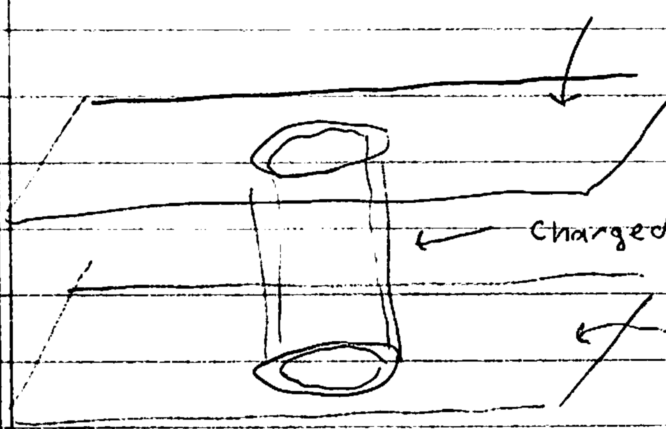
$$\Phi = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right]P_\ell(\cos\theta) \tag{D.5}$$

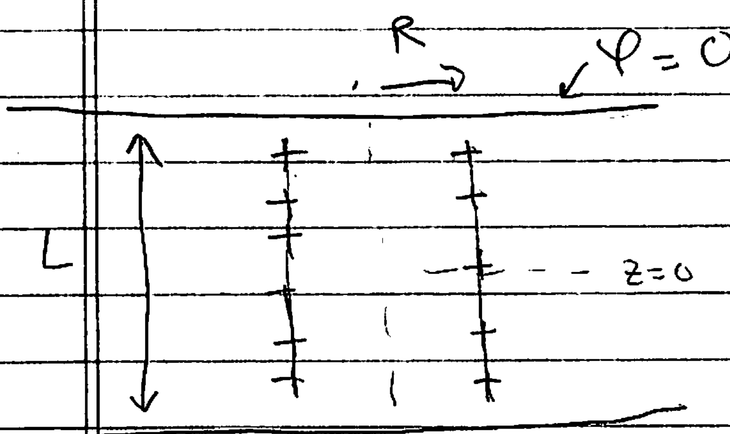
where $P_\ell(\cos\theta)$ is the legendre polynomial, which up to a normalization if $Y_{\ell 0}(\theta, \phi)$, satisfying the orthogonality

$$\int_{-1}^{1} d(\cos\theta)P_\ell(\cos\theta)P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

A worked example pg. 1

Charged Cylinder: $\varphi = 0$ metal grounded plates





with Charge per length λ

Determine $\varphi(\rho, z)$ both inside and outside the cylinder: Concentrate on $z = 0$

Warm up questions:

(1) What are the dimensionfull parametes?

(2) What are the boundary conditions?
- → what is the perpendicular directions
- → What are the parallel directions

(3) What do you qualitatively expect when $L \gg R$ at $z = ($

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Solution: (Qualitative)

① λ, L, R, and z, ρ

at $z=0$ the solution depends on how ρ compares to L, R

② Boundary conditions:

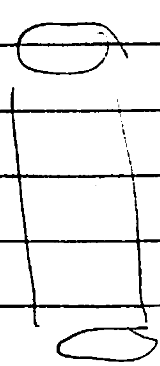
$\varphi = 0$ at $z = \pm L/2$

$$E_\rho \Big|_{R+\varepsilon} - E_\rho \Big|_{R-\varepsilon} = \sigma = \frac{\lambda}{2\pi R} \quad \text{for all } z$$

→ take z, ϕ to be parallel directions and ρ to be perp directions

③ For

$R \ll \rho \ll L$ the walls are infinitely far away, Gauss Law gives:



$$E_\rho = \frac{\lambda}{2\pi\rho}$$

$$\varphi = -\frac{\lambda}{2\pi} \log(\rho) + \text{Const}$$

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Then

- As $\rho \sim L$ start to feel the effect of walls

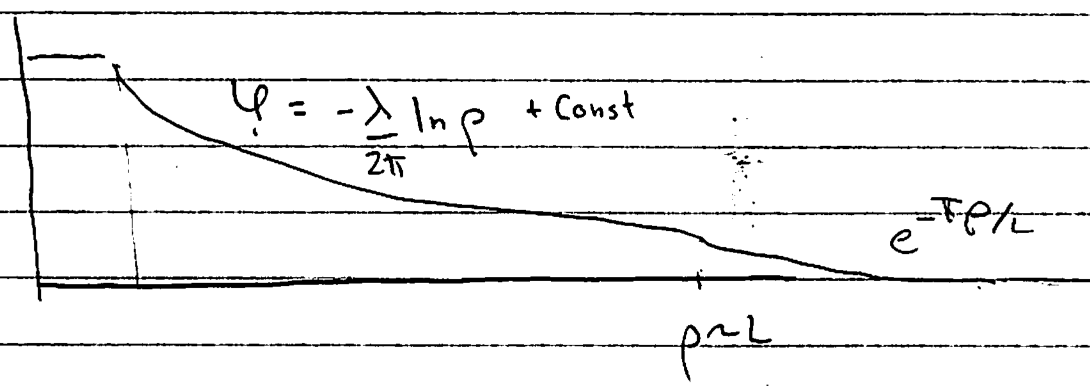
- for $R \ll L \ll \rho$ the fields will decrease exponentially

- Inside the cylinder expect

$$E_\rho = 0$$

$\varphi = \text{const}$ up to corrections suppressed by R^2/L^2

$\varphi(\rho)|_{z=0}$



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Solution (Quantitative)

$$\varphi = \sum_k R_k(\rho)\, Z_k(z)$$

The laplace eqn inside and outside

$$-\nabla^2 \varphi = 0$$

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{-\partial^2}{\partial z^2} \right]\varphi = 0$$

Leads to the separated eqs:

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + k^2 \right] R_k(\rho) = 0$$

$$\left[\frac{-\partial^2}{\partial z^2} - k^2 \right] Z_k(z) = 0$$

- $$Z_k = \underbrace{A_0 + B_0 z}_{\substack{\cancel{k=0} \\ \text{Solution}}} + \underbrace{\sum_k A_k \cos kz + B_k \sin kz}_{\text{general solution}}$$

- Boundary conditions at $z = -L/2$ and $z = L/2$

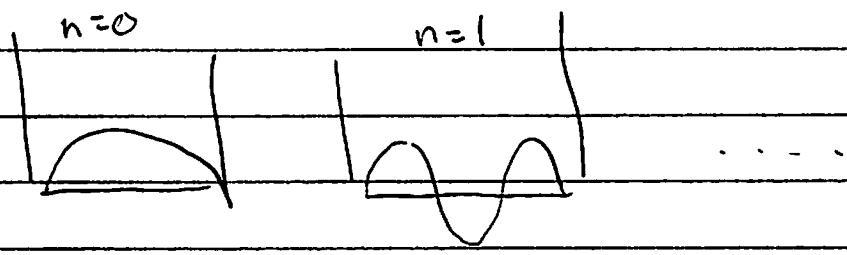
+ symmetry $\varphi(z = -L/2) = \varphi(z = L/2) = 0$

$$Z_k = \sum_n A_k \cos(k_n z)$$

A worked example pg. 4

Now since the function must vanish at $z = -L/2$ and $L/2$

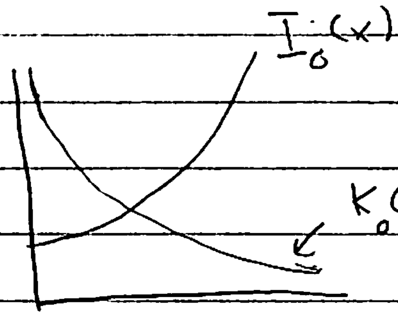
$$k_n = \frac{(2n+1)\pi}{L}$$



From the radial direction

$$R_k(\rho) = A_k I_0(k\rho) + B_k K_0(k\rho)$$

- Asymptotics:



for $x \ll 1$:

$$I_0 = 1 + x^2/4 + \ldots$$

$$K_0 = -\left[\log \frac{x}{2} + \gamma_E\right] I_0 + O(x^2)$$

for $x \gg 1$

$$I_0 = \frac{e^x}{\sqrt{2\pi x}}\left(1 + O\left(\frac{1}{x}\right)\right)$$

$$K_0 = e^{-x}\sqrt{\frac{\pi}{2x}}$$

So inside the cylinder

$$R_k(\rho) = A_k I_0(k\rho)$$

And outside

$$R_k(\rho) = B_k K_0(k\rho)$$

Continuity at $\rho = R$ shows $B_k = I_0(kR)$ $A_k = K_0(k$

$$R_k(\rho) = C_k\Big[K_0(kR)\, I_0(k\rho)\, \Theta(R-\rho) + I_0(kR)\, K_0(k\rho)\, \Theta(\rho - R)\Big]$$

So the solution at this point is

$$\varphi(\rho,z) = \sum_n C_n \left[K_0(kR)\, I_0(k\rho)\, \Theta(R-\rho) + I_0(kR)\, K_0(k\rho)\, \Theta(\rho-R) \right] \times \cos(k_n z)$$

where $k_n = \frac{(2n+1)\pi}{L}$ $n = 0, 1, 2, 3 \ldots$

From the jump condition can determine C_n

$$E_\rho^{out} - E_\rho^{in} = \frac{\lambda}{2\pi R}$$

$$E_\rho^{in} = -\sum_n C_n\, K_0(kR)\, I_0'(kR)\, k_n \cos(k_n z)$$

$$E_\rho^{out} = -\sum_n C_n\, I_0(kR)\, K_0'(kR)\, k_n \cos(k_n z)$$

$$E_\rho^{in}(k) = \frac{2}{L}\int_{-L/2}^{L/2} \cos(k_n z)\, E_\rho^{out}$$

$$= -C_n\, K_0(k_n R)\, I_0'(k_n R)\, k_n$$

$$E_\rho^{out}(k) = -C_n\, I_0(kR)\, K_0'(k_n R)\, k_n$$

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Now

$$E_\rho^{out}(k) - E_\rho^{in}(k) = \frac{2}{L}\int_{-L/2}^{L/2} \frac{\lambda}{2\pi R}\cos(k_n z) \equiv \frac{\lambda(k}{2\pi}$$

$$= \frac{(-1)^n\, 2\lambda}{(1+2n)\pi^2 R}$$

So we get that

$$-C_n \underbrace{\left[I_0(k_n R)\, K_0'(k_n R) - K_0(kR)\, I_0'(kR)\right]}_{\text{Wronsk}} k_n$$

$$= \frac{(-1)^n\, 2\lambda}{(1+2n)\pi^2 R}$$

Recognizing the Wronskian (which often appears in jump conditions, that are derived from Gauss law) we know that this will take a simple form. From bessels eqn:

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + k^2\right]\left(I_0(k\rho) \text{ or } K_0(k\rho)\right) = 0$$

$p(x)$ of Sturm-Louiville eqn $\Rightarrow$ $p(x) = \rho$

We know that $\rho \times \text{Wronsk}(k\rho) = \text{Const}$ ✓

- Use series to show $\text{Wronsk}\Big|_{\rho=R} = \frac{-1}{k\rho}\Big|_{\rho=R} = \frac{-1}{kR}$

So

$$-C_n \left[\frac{-1}{K_n' R} \right] k_n = (-1)^n \frac{2\lambda}{(1+2n)\pi^2 R}$$

So C_n is

$$C_n = (-1)^n \frac{2\lambda}{(1+2n)\pi^2}$$

Thus we have determined the full solution

$$\boxed{\varphi(\rho, z) = \lambda \sum_{n=0}^{\infty} (-1)^n \frac{2}{(1+2n)\pi^2} \times \left[K_0(kR)\, I_0(k\rho)\, \Theta(R-\rho) + I_0(kR)\, K_0(k\rho)\, \Theta(\rho-R) \right] \times \cos(k_n z)}$$

Lets look at $z=0$ and outside

$$\varphi(\rho) = \lambda \sum_{n=0}^{\infty} (-1)^n \frac{2}{(1+2n)\pi^2}\, I_0(kR)\, K_0(k\rho)\, \Theta(\rho-R)$$

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- for $\rho > R$ but $\rho \ll L$ then

$$k_n \rho \simeq \frac{(2n+1)\pi}{L}\rho \ll 1 \quad \text{for almost all } n$$

and $I_0 \simeq 1$ $\quad K_0 \simeq -\ln \frac{k_n \rho}{2} - \gamma_E \simeq -\ln\rho + 2 - \gamma_E + \ln k_n$

$$\varphi(\rho) = \lambda \sum_{n=0}^{\infty} (-1)^n \frac{2}{(1+2n)\pi^2}\left[(-\ln\rho) + \ln k_n + \text{Cons}\right.$$

$$\varphi(\rho) = -\frac{\lambda}{2\pi}\ln\rho + \text{const}$$

this was by construction

we used that $\lambda \sum_{n=0}^{\infty} (-1)^n \frac{2}{(1+2n)\pi^2} = \frac{\lambda}{2\pi}$

So

$$\varphi(\rho) = -\frac{\lambda}{2\pi}\ln\rho + \text{const} \quad \checkmark$$

- for ρ large $\quad R \ll L \ll \rho \quad$ find

$$K_0(k_n\rho) \simeq \frac{1}{\sqrt{2\pi k_n \rho}} e^{-k_n\rho} \qquad k_n = \frac{(2n+1)\pi}{L}\rho$$

- The larger the n the more it is suppressed. keep only the $n=0$ term

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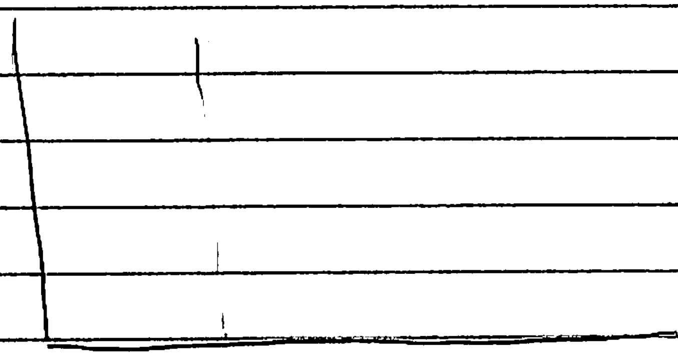
Find outside

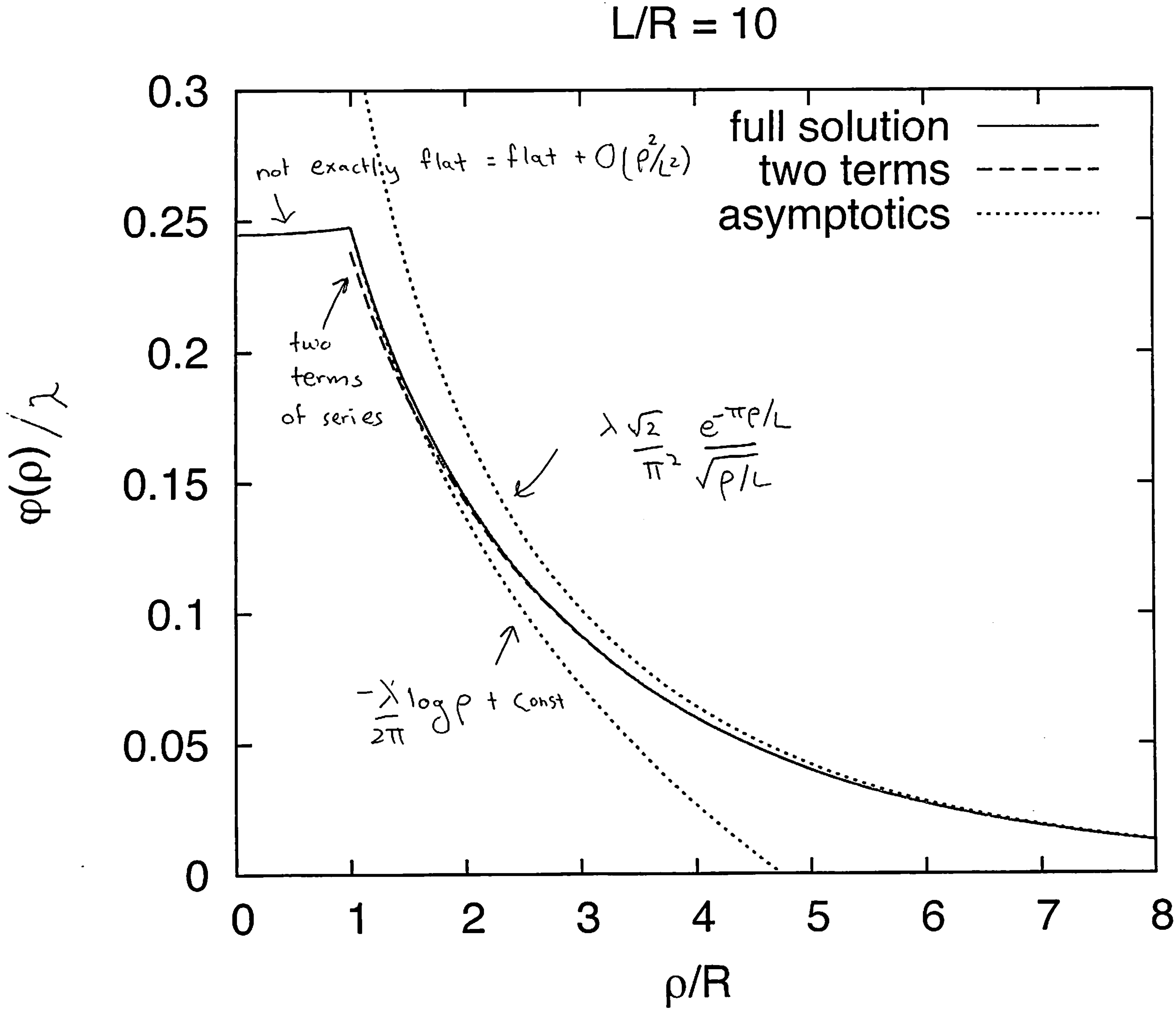
$$\varphi(\rho) = \lambda \sum_{n=0}^{\infty} \frac{(-1)^n\, 2}{(1+2n)\pi^2} \overbrace{I_0(kR)}\, K_0(k\rho)$$

$$\varphi(\rho) \simeq \lambda \frac{2}{\pi^2} e^{-\pi\rho/L} \sqrt{\frac{\pi}{2(\pi\rho/L)}}$$

n=0 only + asympto

$$\varphi(\rho) = \lambda \frac{\sqrt{2}}{\pi^2} \frac{e^{-\pi\rho/L}}{\sqrt{(\rho/L)}}$$




L/R = 10
φ(ρ) / λ
ρ/R
full solution
two terms
asymptotics
not exactly flat = flat + O(ρ²/L²)
two terms of series
λ √2/π² e^{-πρ/L}/√(ρ/L)
-λ/2π log ρ + const
0
0.05
0.1
0.15
0.2
0.25
0.3
0
1
2
3
4
5
6
7
8