11 Radiation in Non-relativistic Systems

11.1 Basic equations

This first section will *NOT* make a non-relativistic approximation, but will examine the far field limit.

(a) We wrote down the wave equations in the covariant gauge:

\[-\Box \Phi = \rho(t_o, r_o)\]  \hspace{1cm} (11.1)

\[-\Box A = J(t_o, r_o)/c\]  \hspace{1cm} (11.2)

The gauge condition reads

\[\frac{1}{c} \partial_t \Phi + \nabla \cdot A = 0\]  \hspace{1cm} (11.3)

(b) Then we used the green function of the wave equation

\[G(t, r\,|\,t_o, r_o) = \frac{1}{4\pi |r - r_o|} \delta(t - t_o + \frac{|r - r_o|}{c})\]  \hspace{1cm} (11.4)

to determine the potentials \((\Phi, A)\)

\[\Phi(t, r) = \int d^3x_o \frac{1}{4\pi |r - r_o|} \rho(T, r_o)\]

\[A(t, r) = \int d^3x_o \frac{1}{4\pi |r - r_o|} J(T, r_o)/c\]  \hspace{1cm} (11.6)

Here \(T(t, r)\) is the retarded time

\[T(t, r) = t - \frac{|r - r_o|}{c}\]  \hspace{1cm} (11.7)

(c) We used the potentials to determine the electric and magnetic fields. Electric and magnetic fields in the far field are

\[A_{\text{rad}}(t, r) = \frac{1}{4\pi r} \int_{r_o} J(T, r_o)/c\]  \hspace{1cm} (11.8)

and

\[B(t, r) = -\frac{n}{c} \times \partial_t A_{\text{rad}}\]

\[E(t, r) = n \times \frac{n}{c} \times \partial_t A_{\text{rad}} = -n \times B(t, r)\]  \hspace{1cm} (11.10)

In the far field (large distance limit \(r \to \infty\)) limit we have

\[T = t - \frac{r}{c} + \frac{n \cdot r_o}{c}\]  \hspace{1cm} (11.11)
And we recording the derivatives

\[
\left( \frac{\partial}{\partial t} \right)_{\mathbf{r}_o} = \left( \frac{\partial}{\partial T} \right)_{\mathbf{r}_o} \\
\left( \frac{\partial}{\partial \mathbf{r}_o} \right)_{t} = \left( \frac{\partial}{\partial \mathbf{T}} \right)_{\mathbf{r}_o} + \frac{n}{c} \left( \frac{\partial}{\partial T} \right)_{\mathbf{r}_o}
\]

(11.12) (11.13)

(d) We see that the radiation (electric field) is proportional to the transverse piece of the \( \partial_t \mathbf{J} \)

\[- \mathbf{n} \times (\mathbf{n} \times \partial_t \mathbf{J}) = \partial_t \mathbf{J} - \mathbf{n} (\mathbf{n} \cdot \partial_t \mathbf{J})
\]

(11.14)

In general the transverse projection of a vector is

\[- \mathbf{n} \times (\mathbf{n} \times \mathbf{V}) = \mathbf{V} - \mathbf{n} (\mathbf{n} \cdot \mathbf{V})
\]

(11.15)

(e) Power radiated per solid angle is for \( r \to \infty \) is

\[
\frac{dW}{dt d\Omega} = \frac{dP(t)}{d\Omega} = \text{energy per observation time per solid angle}
\]

(11.16)

and

\[
\frac{dP(t)}{d\Omega} = r^2 \mathbf{S} \cdot \mathbf{n} \]

(11.17)

\[
= c |r \mathbf{E}|^2
\]

(11.18)

11.2 Examples of Non-relativistic Radiation: L31

In this section we will derive several examples of radiation in non-relativistic systems. In a non-relativistic approximation

\[
T = t - \frac{r}{c} + \frac{\mathbf{n} \cdot \mathbf{r}_o}{\text{small}}
\]

(11.19)

The underlined terms are small: If the typical time and size scales of the source are \( T_{\text{typ}} \) and \( L_{\text{typ}} \), then \( t \sim T_{\text{typ}} \), and \( \mathbf{r}_o \sim L_{\text{typ}} \), and the ratio the underlined term to the leading term is:

\[
\frac{L_{\text{typ}}}{c T_{\text{typ}}} \ll 1
\]

(11.20)

This is the non-relativistic approximation. For a harmonic time dependence, \( 1/T_{\text{typ}} \sim \omega_{\text{typ}} \), and this says that the wave number \( k = \frac{2\pi}{\lambda} \) is small compared to the size of the source, i.e. the wave length of the emitted light is long compared to the size of the system in non-relativistic motion:

\[
\frac{2\pi L_{\text{typ}}}{\lambda} \ll 1
\]

(11.21)

(a) Keeping only \( t - r/c \) and dropping all powers of \( \mathbf{n} \cdot \mathbf{r}_o / c \) in \( T \) results in the electric dipole approximation, and also the Larmour formula.

(b) Keeping the first order terms in

\[
\frac{\mathbf{n}}{c} \cdot \mathbf{r}_o
\]

(11.22)

results in the magnetic dipole and quadrupole approximations.
The Larmour Formula

(a) For a particle moves slowly with velocity and acceleration, \( \mathbf{v}(t) \) and \( \mathbf{a}(t) \) along a trajectory \( \mathbf{r}_*(t) \)

(b) We make an ultimate non-relativistic approximation for \( T \)

\[
T \simeq t - \frac{r}{c} = t_e
\]

Then we derived the radiation field by substituting the current

\[
\mathbf{J}(t_e) = e\mathbf{v}(t_e)\delta^3(\mathbf{r}_o - \mathbf{r}_*(t_e))
\]

into the Eqs. (11.8), (11.9), and (11.17) for the radiated power

(c) The electric field is

\[
\mathbf{E} = \frac{e}{4\pi rc^2}\mathbf{n} \times \mathbf{n} \times \mathbf{a}(t_e)
\]

Notice that the electric field is of order

\[
E \sim \frac{e}{4\pi r} \frac{a(t_e)}{c^2}
\]

(d) The power per solid angle emitted by acceleration at time \( t_e \) is

\[
\frac{dP(t_e)}{d\Omega} = \frac{e^2}{(4\pi)^2c^3}a^2(t_e)\sin^2 \theta
\]

Notice that the power is of order

\[
P \sim c|\mathbf{rE}|^2 \sim \frac{a^2}{c^3}
\]

(e) The total energy that is emitted is

\[
P(t_e) = \frac{e^2}{4\pi} \frac{2a^2(t_e)}{3c^3}
\]

The Electric Dipole approximation

(a) We make the ultimate non-relativistic approximation

\[
\mathbf{J}(t - \frac{r}{c} + \frac{n \cdot \mathbf{r}_o}{c}) \simeq \mathbf{J}(t - \frac{r}{c})
\]

Leading to an expression for \( \mathbf{A}_{rad} \)

\[
\mathbf{A}_{rad} = \frac{1}{4\pi rc} \frac{1}{c}\partial_t \mathbf{p}(t_e)
\]

where the dipole moment is

\[
\mathbf{p}(t_e) = \int d^3x_o \rho(t_e)\mathbf{r}_o
\]

(b) The electric and magnetic fields are

\[
\mathbf{E}_{rad} = \mathbf{n} \times \mathbf{n} \times \frac{1}{c}\partial_t \mathbf{A}_{rad}
\]

\[
= \frac{1}{4\pi rc^2} \mathbf{n} \times \mathbf{n} \times \dot{\mathbf{p}}(t_e)
\]

\[
\mathbf{B}_{rad} = \mathbf{n} \times \mathbf{E}_{rad}
\]

(c) The power radiated is

\[
\frac{dP(t_e)}{d\Omega} = \frac{1}{16\pi^2} \frac{\omega^2(t_e)}{c^3} \sin^2 \theta
\]

(d) For a harmonic source \( \mathbf{p}(t_e) = \mathbf{p}_0 e^{-i\omega(t-r/c)} \) the time averaged power is

\[
P = \frac{1}{4\pi} \frac{\omega^4}{3c^3} |\mathbf{p}_0|^2
\]
The magnetic dipole and quadrupole approximation: L32

(a) In the magnetic dipole and quadrupole approximation we expand the current

\[ J(T) \approx J(t_e) + \frac{n \cdot r_o}{c} \partial_t J(t_e, r_o) / c \]  

The next term when substituted into Eq. (11.8) gives rise to new contributions to \( \mathbf{A}_{\text{rad}} \), the magnetic dipole and electric quadrupole terms:

\[ \mathbf{A}_{\text{rad}} = \mathbf{A}_{\text{rad}}^{E1} + \mathbf{A}_{\text{rad}}^{M1} + \mathbf{A}_{\text{rad}}^{E2} \]  

(b) The magnetic dipole contribution gives

\[ A_{\text{rad}}^{M1} = -\frac{1}{4\pi} \frac{n}{c} \times \dot{m}(t_e) \]  

where \( \mathbf{m} \)

\[ \mathbf{m} \equiv \frac{1}{2} \int_{r_o} r_o \times J(t_e, r_o) / c, \]  

is the magnetic dipole moment.

(c) The structure of magnetic dipole radiation is very similar to electric dipole radiation with the duality transformation

<table>
<thead>
<tr>
<th>E-dipole</th>
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<th>M-dipole</th>
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<tbody>
<tr>
<td>( p )</td>
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<td>( m )</td>
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<tr>
<td>( E )</td>
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<td>( B )</td>
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<td>(-E)</td>
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</tbody>
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(d) The power is

\[ \frac{dP^{M1}(t_e)}{d\Omega} = \frac{\ddot{m}^2 \sin^2 \theta}{16\pi^2 c^3} \]  

(e) The power radiated in magnetic dipole radiation is smaller than the power radiated in electric dipole radiation by a factor of the typical velocity, \( v_{\text{typ}} \) squared:

\[ \frac{P^{M1}}{P^{E1}} \propto \frac{n^2}{p^2} \sim \left( \frac{v_{\text{typ}}}{c} \right)^2 \]  

where \( v_{\text{typ}} \sim L_{\text{typ}}/T_{\text{typ}} \)

Quadrupole radiation

(a) For quadrupole radiation we have

\[ A_{\text{rad},E2} = \frac{1}{24\pi} \frac{n_i}{c^2} Q^{ij} \]  

where \( Q^{ij} \) is the symmetric traceless quadrupole tensor.

\[ Q^{ij} = \int d^3 x_o \rho(t_e, r_o) (3r_o^i r_o^j - r_o^2 \delta^{ij}) \]
(b) The electric field is
\[ E_{\text{rad}} = -\frac{1}{24\pi rc^3} \left[ \vec{Q} \cdot \vec{n} - \vec{n}(\vec{n} \cdot \vec{Q} \cdot \vec{n}) \right] \] (11.50)
where (more precisely) the first term in square brackets means \( n_i \bar{Q}^{ij} \), while the second term means, \( (n_i \bar{Q}^{lm} p_m) n^j \).

(c) A fair bit of algebra shows that the total power radiated from a quadrupole form is
\[ P = \frac{1}{720\pi c^5} \bar{Q}^{ab} \bar{Q}_{ab} \] (11.51)

(d) For harmonic fields, \( Q = Q_o e^{-i\omega t} \), the time averaged power is rises as \( \omega^6 \)
\[ P = \frac{c}{1440\pi} \left( \frac{\omega}{c} \right)^6 Q_o^2 \] (11.52)

(e) The total power radiated radiated in quadrupole radiation to electric-dipole radiation for a typical source size \( L_{\text{typ}} \) is smaller:
\[ \frac{P_{E2}}{P_{E1}} \sim \left( \frac{\omega L_{\text{typ}}}{c} \right)^2 \] (11.53)

11.3 Attenas

(a) In an antenna with sinusoidal frequency we have
\[ J(T, r_o) = e^{-i\omega(t-r/c + \vec{n} \cdot \vec{r}_o)} J(r_o) \] (11.54)

(b) Then the radiation field for a sinusoidal current is:
\[ A_{\text{rad}} = \frac{e^{-i\omega(t-r/c)}}{4\pi r} \int_{r_o} e^{-i\omega \frac{\vec{n} \cdot \vec{r}_o}{r}} J(r_o)/c \] (11.55)

In general one will need to do this integral to determine the radiation field.

(c) The typical radiation resistance associated with driving a current which will radiate over a wide range of frequencies is \( R_{\text{vacuum}} = c\mu_o = \sqrt{\mu_o/\epsilon_o} = 376 \text{ Ohm} \).