# 5 Ohms Law and Conduction

## 5.1 Steady current and Ohms Law: Lecture 17

(a) For steady currents

$$\nabla \cdot \boldsymbol{j} = 0 \tag{5.1}$$

(b) For steady currents in ohmic matter

$$\boldsymbol{j} = \sigma \boldsymbol{E} \tag{5.2}$$

(c) $\sigma$ has units of $1/s$. Note that in MKS units $\sigma_{MKS}$ has the uninformative unit 1/ohm m:

$$\sigma_{HL} = \frac{\sigma_{MKS}}{\varepsilon_o} \tag{5.3}$$

For $\sigma_{MKS} = 10^7\,(\text{ohm m})^{-1}$ we find $\sigma \sim 10^{18}\,\text{s}^{-1}$.

(d) To find the flow of current we need to solve the electrostatics problem

$$-\nabla \cdot (\sigma \boldsymbol{E}) = 0 \tag{5.4}$$

$$\nabla \times \boldsymbol{E} = 0 \tag{5.5}$$

or for homogeneous material ($\sigma = \text{const}$)

$$-\sigma \nabla^2 \Phi = 0 \tag{5.6}$$

We see that we are supposed to solve the Laplace equation. However the boundary conditions are rather different.

(e) A point source of current is represented by a delta function $I\delta^3(\boldsymbol{r} - \boldsymbol{r}_o)$. While a sink of current is represented by a delta function of opposite sign $-I\delta^3(\boldsymbol{r} - \boldsymbol{r}_o)$.

(f) Eq. (5.4) and Eq. (5.6) need boundary conditions. At an interface current should be conserved so

$$\boldsymbol{n} \cdot (\boldsymbol{j}_2 - \boldsymbol{j}_1) = 0 \tag{5.7}$$

or

$$\sigma_2 \frac{\partial \Phi_2}{\partial n} = \sigma_1 \frac{\partial \Phi_1}{\partial n} \tag{5.8}$$

Most often this is used to say that the normal component of the Electric field at a metal-insulator interface should be zero:

$$\boldsymbol{n} \cdot \boldsymbol{E} = 0 \qquad \text{at metal-insulator interface} \tag{5.9}$$

(g) In general the input current (or normal derivatives of the potential) must be specified at all the boundaries in order to have a well posed boundary value problem that can be solved (at least numerically.)

(h) In general the input currents $I_a = I_1, I_2, \ldots$ on a set conductors will be will be specified, specifying the normal derivatives on all of the surfaces. Then you solve for the potential. The voltages of a given electrode relative to ground is $V_a$ , and you will find that $V_a = \sum_b R_{ab} I_b$. $R_{ab}$ is the resistance matrix.

# 6 Magneto Statics and Magnetic Matter

## 6.1 Magneto-Statics

At first order in $1/c$ we have the magneto static equations

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}_{tot}}{c} \qquad \boldsymbol{j}_{\rm tot} = \frac{\boldsymbol{j}}{c} + \underbrace{\frac{1}{c}\partial_t \boldsymbol{E}^{(0)}}_{\text{displacement current}} \tag{6.1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{6.2}$$

where $\boldsymbol{j}_D = 1/c\,\partial_t \boldsymbol{E}^{(0)}$ is the displacement current. The formulas given below assume that $\boldsymbol{j}_D$ is zero. But, with no exceptions apply if one replaces $\boldsymbol{j} \to \boldsymbol{j} + \boldsymbol{j}_D$.

The current is taken to be steady

$$\nabla \cdot \boldsymbol{j} = 0 \tag{6.3}$$

### Computing Fields: Lecture 14 and 15

(a) Below we note that for a current carrying wire

$$\boldsymbol{j}\mathrm{d}^3x = I\mathrm{d}\boldsymbol{\ell} \tag{6.4}$$

(b) We can compute the fields using the integral form of Ampères law $\nabla \times \boldsymbol{B} = j/c$, which says that the loop integral of $\boldsymbol{B}$ is equal to the current piercing the area bounded by the loop

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{\ell} = \frac{I_{\rm pierce}}{c} \tag{6.5}$$

For the familiar case of a current carrying wire we found $B_\phi = (I/c)/2\pi\rho$, where $\rho$ is the distance from the wire.

(c) The Biot-Savat Law is seemingly similar to the coulomb law

$$\boldsymbol{B}(\boldsymbol{r}) = \int \mathrm{d}^3x_o \, \frac{\boldsymbol{j}(\boldsymbol{r}_o)/c \times \widehat{\boldsymbol{r} - \boldsymbol{r}_o}}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|^2} \tag{6.6}$$

We used this to compute the magnetic field of a ring of radius on the z-axis

$$B_z = 2\frac{(I/c)\pi a^2}{4\pi\sqrt{z^2 + a^2}} \tag{6.7}$$

which you can remember by knowing magnetic moment of the ring and other facts about magnetic dipoles (see below)

(d) Using the fact that $\nabla \cdot \boldsymbol{B} = 0$ we can write it as the curl of $\boldsymbol{A}$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\Lambda \tag{6.8}$$

but recognize that we can always add a gradient of a scalar function $\Lambda$ to $\boldsymbol{A}$ without changing $\boldsymbol{B}$.

(e) If we adopt the coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$ and use the much used identity

$$\nabla \times (\nabla \times A) = -\nabla^2 \boldsymbol{A} + \boldsymbol{\nabla}(\nabla \cdot \boldsymbol{A}) \,, \tag{6.9}$$

we get the result

$$-\nabla^2 \boldsymbol{A} = \frac{\boldsymbol{j}}{c} \,. \tag{6.10}$$

Then in free space $\boldsymbol{A}$ satisfies

$$\boldsymbol{A}(\boldsymbol{r}) = \int \mathrm{d}^3 x_o \frac{\boldsymbol{j}(\boldsymbol{r}_o)/c}{4\pi |\boldsymbol{r} - \boldsymbol{r}_o|} \tag{6.11}$$

(f) The equations must be supplemented by boundary conditions. In vacuum we have that the parallel components of $\boldsymbol{B}$ jump according to size of the surface currents $\boldsymbol{K}$, while the normal components of $\boldsymbol{B}$ are continuous

$$\boldsymbol{n} \times (\boldsymbol{B}_2 - \boldsymbol{B}_1) = \frac{\boldsymbol{K}}{c} \tag{6.12}$$

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{6.13}$$

Here $\boldsymbol{K}$ is the surface current and has units charge/length/s.

### Multipole expansion of magnetic fields: Lecture 16

We wish to compute the magnetic field far from a localized set of currents. We can start with Eq. (6.14) and determine that far from the sources the vector potential is described by the magnetic dipole moment:

(a) The vector potential is

$$\boldsymbol{A} = \frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{4\pi r^2} \tag{6.14}$$

where

$$\boldsymbol{m} = \tfrac{1}{2} \int \mathrm{d}^3 x_o \boldsymbol{r}_o \times \boldsymbol{j}(\boldsymbol{r}_o)/c \tag{6.15}$$

is the magnetic dipole moment.

(b) For a current carrying wire:

$$\boldsymbol{m} = \frac{I}{c} \frac{1}{2} \oint \boldsymbol{r}_o \times \mathrm{d}\boldsymbol{\ell}_o = \frac{I}{c} \boldsymbol{a} \tag{6.16}$$

(c) The magnetic field from a dipole

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{3(\boldsymbol{n} \cdot \boldsymbol{m}) - \boldsymbol{m}}{4\pi r^3} \tag{6.17}$$

(d) **U**NITS NOTE: I defined $\boldsymbol{m}$ in Eq. (6.15) with $\boldsymbol{j}/c$. This has the "feature" that that

$$\boldsymbol{m}_{HL} = \frac{\boldsymbol{m}_{MKS}}{c} \tag{6.18}$$

In MKS units

$$\boldsymbol{A}_{MKS} = \mu_o \frac{\boldsymbol{m}_{MKS} \times \hat{\boldsymbol{r}}}{4\pi r^2} \tag{6.19}$$

Setting $\varepsilon_o = 1$ so $\mu_o = 1/c^2$ and multiplying by $c$

$$\boldsymbol{A}_{HL} = c\boldsymbol{A}_{MKS} = \frac{\boldsymbol{m}_{MKS}/c \times \hat{\boldsymbol{r}}}{4\pi r^2} = \frac{\boldsymbol{m}_{HL} \times \hat{\boldsymbol{r}}}{4\pi r^2} \tag{6.20}$$

Below we will define the magnetization, and similarly $\boldsymbol{M}_{HL} = \boldsymbol{M}_{MKS}/c$.

### Forces on currents

(a) We wish to compute the force on a small current carrying object in an external magnetic field. For a compact region of current (which is small compared to the inverse gradients of the external magnetic field) the total magnetic force is

$$\boldsymbol{F}(\boldsymbol{r}_o) = (\boldsymbol{m}\cdot\nabla)\,\boldsymbol{B}(\boldsymbol{r}_o) \tag{6.21}$$

where $\boldsymbol{m}$ is measured with respect $\boldsymbol{r}_o$, *i.e.*

$$\boldsymbol{m} = \frac{1}{2}\int_V \mathrm{d}^3x\ \delta\boldsymbol{r}\times\boldsymbol{j}(\boldsymbol{r})/c \tag{6.22}$$

with $\delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_o$.

(b) For a fixed dipole magnitude we have $\boldsymbol{F} = \nabla(\boldsymbol{m}\cdot\boldsymbol{B})$ or

$$U(\boldsymbol{r}_o) = -\boldsymbol{m}\cdot\boldsymbol{B}(\boldsymbol{r}_o) \tag{6.23}$$

This formula is the same as the MKS one since we have taken $\boldsymbol{m}_{HL} = \boldsymbol{m}_{MKS}/c$.

(c) The torque is

$$\boldsymbol{\tau} = \boldsymbol{m}\times\boldsymbol{B} \tag{6.24}$$

(d) Finally (we will discuss this later) the magnetic force on a current carrying region is

$$(\boldsymbol{F}_B)^j = \frac{1}{c}\int_V (\boldsymbol{j}\times\boldsymbol{B})^j = -\int_{\partial V} dS\,\boldsymbol{n}_i T_B^{ij} \tag{6.25}$$

where

$$T_B^{ij} = -B^iB^j + \frac{1}{2}\boldsymbol{B}^2\delta^{ij} \tag{6.26}$$

is the magnetic stress tensor and $\boldsymbol{n}$ is an outward directed normal.

### Solving for magneto-static fields

(a) One approach is to use direct integration:

$$\boldsymbol{A}(\boldsymbol{r}) = \mu\int \mathrm{d}^3x_o \frac{\boldsymbol{j}(\boldsymbol{r}_o)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}$$

Then for any current distribution once can compute the magnetic field – see lecture for an example of a rotating charged sphere . This is analogous to using the coulomb law.

(b) Another approach is to view

$$-\nabla^2\boldsymbol{A} = \mu\frac{\boldsymbol{j}}{c} \tag{6.27}$$

as a differential equation and to try separation of variables. There are (at least) two cases where the equations for $\boldsymbol{A}$ simplify.

i) If the current is azimuthally symmetric then it is reasonable to try a form $A_\phi(r,\theta)$

$$-\nabla^2\boldsymbol{A} = \mu\frac{\boldsymbol{j}}{c} \Rightarrow -\nabla^2 A_\phi + \frac{A_\phi}{r^2\sin^2\theta} = \mu\frac{j_\phi}{c} \tag{6.28}$$

Here the $-\nabla^2 A_\phi$ is the scalar Laplacian in spherical coordinates. For instance, this is an effective way to find the magnetic field from a ring of current or a rotating charged sphere.

ii) If the current runs up and down then you can try $A_z(\rho,\phi)$ in cylindrical coordinates:

$$-\nabla^2 A_z(\rho,\phi) = \mu\frac{j_z}{c} \tag{6.29}$$

Here $\nabla^2 A_z$ is the scalar Laplacian in cylindrical coordinates. See homework for an example of a cylindrical shell.

(c) Finally if the current separates two (or more) distinct regions of space (such as in a rotating charged sphere), then in each region one has

$$\nabla \times \boldsymbol{H} = 0 \tag{6.30}$$

So for each region one can introduce a scalar potential $\psi_m$ such that

$$\boldsymbol{H} = -\nabla \psi_m \tag{6.31}$$

and (using $\nabla \cdot \boldsymbol{B} = 0$) show that

$$-\mu \nabla^2 \psi_m = 0 \tag{6.32}$$

assuming $\mu$ is constant. Then the Laplace equation is solved in each region, and the boundary conditions (Eq. (6.49)) are used to connect the scalar potential across regions. The boundary conditions are markedly different from the electrostatic case, and this leads to markedly different solutions. See lecture for an example of the magnetic moment induced by an external field.