Boundary Conditions

Now we want to solve

\[ \nabla \cdot \mathbf{D} = \rho_{\text{ext}} \]
\[ \nabla \times \mathbf{E} = 0 \]

Before we can do so we need boundary conditions

From \( \nabla \cdot \mathbf{D} = \rho_{\text{ext}} \), we apply Gauss law to this and find

\[ \mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_{\text{ext}} \]

i.e. the normal components of \( \mathbf{D} \) jump across the surface

From \( \nabla \times \mathbf{E} = 0 \) or \( \oint \mathbf{E} \cdot d\mathbf{l} = (E'' - E') \ell = 0 \)

we have the continuity of parallel electric field component

\[ E'' = E' \]
A model problem. A spherical cavity (Jackson 4.4)

Consider a cavity in a dielectric of radius $\hat{a}$ in an external field.

The polarized charge we will find is sketched. (Why does it look like this?)

Solution

$$\nabla \cdot D = \rho$$ and for constant $\varepsilon$ $$\nabla \cdot (\varepsilon \nabla \psi) = \rho$$

$$\nabla \times E = 0 \Rightarrow E = -\nabla \psi$$

So outside the sphere $\rho = 0$ and $-\nabla^2 \psi = 0$

$$\psi_{\text{out}} = \sum_{l} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l (\cos \theta)$$

while inside

$$\psi_{\text{in}} = \sum_{l} \left( C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l (\cos \theta)$$

The only non-vanishing $A_l$ is $A_1$.

$$\psi_{\text{out}} = -E_0 r \cos \theta + \sum_{l} \frac{B_l}{r^{l+1}} \cos \theta$$
Now on surface $r=a$ we have:

\[ \mathbf{n} \cdot \mathbf{D}^{\text{out}} - \mathbf{n} \cdot \mathbf{D}^{\text{in}} = 0 \quad \Rightarrow \quad \frac{\varepsilon \partial \varphi^{\text{out}}}{\partial r} = \frac{\partial \varphi^{\text{in}}}{\partial r} \quad (1) \]

\[ E_2'' - E_1'' = 0 \quad \Rightarrow \quad \frac{\partial^2 \varphi^{\text{out}}}{\partial \theta^2} = \frac{\partial^2 \varphi^{\text{in}}}{\partial \theta^2} \quad (2) \]

In general these equations are easily satisfied for $l \neq 1$. Just set $B_2 = C_2 = 0$. The $l=1$ equation gives a non-trivial condition.

From (1) and (2) we find:

\[ (-\varepsilon E_0 - 2 \varepsilon B_1) = C_1 \frac{1}{a^3} \]

\[ (-E_0 \alpha + B_1) = C_1 \alpha \]

Solving for $C_1$ and $B_1$, we have: (1-ε) is negative

\[ \Phi = \begin{cases} 
- E_0 r \cos \theta + E_0 a^3 \left( 1 - \varepsilon \right) \cos \theta & r > a \\
\frac{1}{\Gamma^2} \left( 1 + 2\varepsilon \right) & r < a \\
- E_0 r \cos \theta \left( \frac{3\varepsilon}{1 + 2\varepsilon} \right) & r < a 
\end{cases} \]
We make the following remarks:

1. The electric field is constant in the sphere.

2. The induced charge is:

$$\mathbf{\sigma}_p = -\mathbf{n} \cdot (\mathbf{p}_2 - \mathbf{p}_1)$$

$$= -\varepsilon \mathbf{n} \cdot \mathbf{E}$$

$$= -(\varepsilon - 1) \left( \frac{\partial \varphi}{\partial r} \right)_{r=a}$$

This yields after differentiating \(\varphi_{out}\):

$$\mathbf{\sigma}_p = -3(\varepsilon - 1) \frac{E_0 \cos \Theta}{1 + 2\varepsilon}$$

Thus we find the appropriate sign.