Energy and Forces in Dielectrics

\[ Q_f + SQ_p \]

See Griffiths (sec 4.4.4) for a clear discussion

\[ Q_f \]

The energy required to add \( dQ_f \) is

\[ SW = \int_V \delta p_f \cdot \nabla \phi \leftarrow \text{External work required to charge capacitor} \]

\[ SW = \int_V \nabla \cdot (\mathbf{S} \mathbf{D}) \phi \leftarrow \text{including to polarize the dielectric} \]

\[ = -\int_V \mathbf{S} \mathbf{D} \cdot \nabla \phi \]

So

\[ SW = \int_V \mathbf{E} \cdot \mathbf{S} \mathbf{D} \mathbf{(E)} \text{ and } W = \int_V \int_D \mathbf{E}^2 dD \]

For linear substance \( \mathbf{S} \mathbf{D} = \varepsilon(r) \mathbf{E} \) and

\[ SW = \int_V \mathbf{E} \cdot \varepsilon(r) \delta \mathbf{E} \text{ and } W = \int_V \frac{1}{2} \varepsilon(r) \mathbf{E}^2(r) \]
So then

\[ W = \frac{1}{2} \int E \cdot D \]

**Force and Stress**

- Focus on material with \( E = \text{constant} \). You can treat piecewise constant materials in the same way, and the answer for the stress holds quite generally.

- The force per volume is (on the free charge)

\[ f^i = \rho_f E^i \]

- Then use \( \rho_f = \mathbf{D} \cdot \mathbf{D} \) and manipulate. The set of steps should be familiar from our previous work on stress.

\[ f^i = -\mathbf{D} \cdot \mathbf{T}^i \]

Where for linear media

\[ T^i = -D^i E^j e_j + \frac{1}{2} D_i \frac{\partial E}{\partial x_j} g^{ij} \]

The proof is given at the end of this section.
As before we can integrate to find the total force on a given volume.

\[ F \hat{j} = \int d^3 r \ f \ hat{j} = -\int d\alpha n_i \ T_i \hat{j} \]

\( \forall \ \Omega \)
Problem

\[ \Phi = V_0 \]
\[ \Phi = 0 \]

**Dielectric** \( \varepsilon_2 \)

**Dielectric** \( \varepsilon_1 \)

\[ -Q_f \]

- Calculate the force per area on the interface. Method 1: energy. Method 2: stress tensor

\[ F^2 = -\int_{\text{dosed line}} dS \cdot n \cdot T_{ij} \]

\[ = -A T_{22}^{22} + A T_{11}^{22} \]

Now

\[ -T_{22}^{22} = \frac{E^2}{\varepsilon_2} D_2^2 - \frac{1}{2} E \cdot D S_{22}^{22} = + \frac{E^2}{\varepsilon_2} D_2^2 = \frac{1}{2} \left( D_2^2 \right)^2 \]

and similarly,

\[ T_{11}^{22} = \frac{1}{2} \left( D_1^2 \right)^2 \]

So

\[ F^2 = \frac{1}{2} \left( D_2^2 \right)^2 - \frac{1}{2} \left( D_1^2 \right)^2 \]

\[ = \frac{1}{2} \frac{\sigma_f^2}{\varepsilon_2} \left( \frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1} \right) \]

Using continuity at the interface, \( D_2 - D_1 = \sigma_f^2 A \)

at the metal plates, \( \vec{n} \cdot \vec{B} = \sigma_f^2 \). Yielding
Problem

Consider the following parallel plate capacitor held

\[ \psi = V_0 \]

\[ \psi = 0 \]

Use the stress tensor to determine the force on the dielectric

Solution

First note that the potential is constant so the electric field is the same at \( V_0 \) \( E_{\text{in}} \) and \( E_{\text{out}} \). The charge density is not the same. In particular...

\[ \sigma_{\text{in}} = D_{\text{in}} = \varepsilon E_{\text{in}} = \varepsilon E_{\text{out}} \text{, while } \sigma_{\text{out}} = E_{\text{out}} \]

Then using the stress tensor on the dashed volume shown

\[ F^x = - \int \text{d}A \, n_i T^{ix} \]

\[ F^x = - A \left( T^{xx}_{\text{out}} - T^{xx}_{\text{in}} \right) \text{ area of side face} \]
The stress tensor on the outside

\[ T_{\text{out}}^{xx} = -E_{\text{out}}^{x}E_{\text{out}}^{x} + \frac{1}{2}E_{\text{out}}^{2}\delta_{xx} \]

\[ T_{\text{out}}^{xx} = \frac{1}{2}E_{\text{out}}^{2} \]

while inside

\[ T_{\text{in}}^{xx} = -E_{\text{in}}^{x}E_{\text{in}}^{x} + \frac{1}{2}E_{\text{in}}^{2}\delta_{xx} \]

\[ = \frac{1}{2}E_{\text{in}}^{2}\varepsilon \]

So the force per area

\[ F^{x} = -\frac{1}{2}E_{\text{out}}^{2} + \frac{1}{2}\varepsilon E_{\text{in}}^{2} \]

\[ = \frac{1}{2}E_{\text{out}}^{2}(\varepsilon - 1) \]

\[ F^{x} = \frac{1}{2}\varepsilon_{0}^{2}(\varepsilon - 1) \]

\[ \frac{F^{x}}{A_{\text{side}}} = \frac{1}{2}\varepsilon_{0}^{2}(\varepsilon - 1) \]