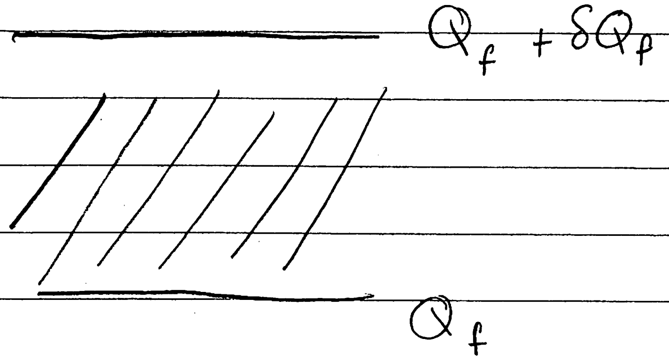
Energy and Forces in Dielectrics



See Griffiths (sec 4.4.4) for a clear discussion

The energy required to add dQ_f is

$$\delta W = \int_V \delta\rho_f \cdot \varphi$$

$\leftarrow$ External wok required to charge capacitor including to polarize the dielectric

$$\delta W = \int_V \nabla \cdot (\delta \vec{D})\, \varphi$$

$$= -\int_V \delta D \cdot \nabla \varphi$$

So

$$\delta W = \int_V \vec{E} \cdot \delta D(\vec{E}) \quad \text{and} \quad W = \int_V \int_0^D \vec{E}(\vec{D}) \cdot d\vec{D}$$

For linear substance $\delta\vec{D} = \varepsilon(r)\,\vec{E}$ and

$$\delta W = \int_V E \cdot \varepsilon(r)\, \delta E \quad \text{and} \quad W = \int_V \frac{1}{2} \varepsilon(r)\, \vec{E}^2(r)$$

So then

$$\boxed{W = \frac{1}{2}\int \vec{E}\cdot\vec{D}}$$

Force and Stress

- Focus on material with ε = constant. You can treat piecewise constant materials in the same way, and the answer for the stress holds quite generally
- The force per volume is (on the free charge)

$$\boxed{f^{j} = \rho_f E^{j}}$$

- Then use $\rho_f = \partial_i D^i$ and manipulate. The set of steps should be familiar from our previous work on stress.

$$\boxed{f^{j} = -\partial_i T^{ij}_{E}}$$

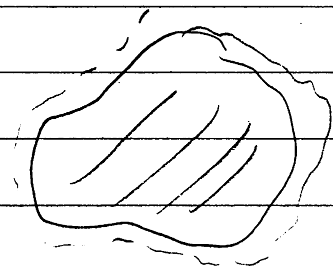
Where for linear media

$$\boxed{T^{ij}_{E} = -D^i E^j + \frac{1}{2}\vec{D}\cdot\vec{E}\,\delta^{ij}}$$

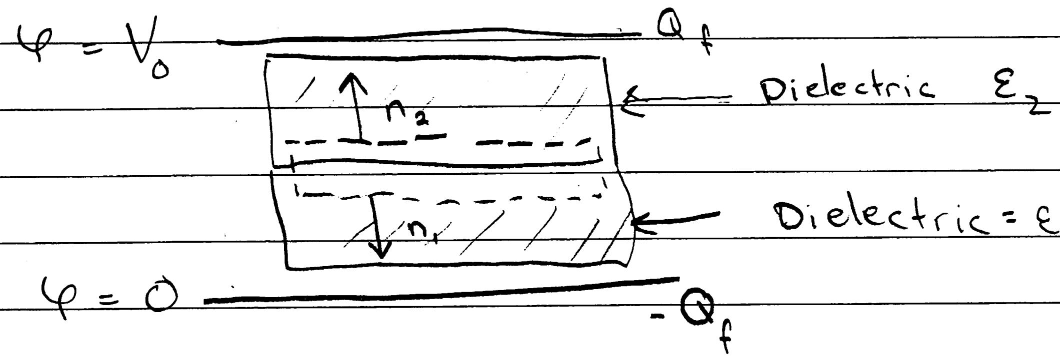
The proof is given at the end of this section

- As before we can integrate to find the total force on a given volume

$$\boxed{F^j = \int_V d^3 r_0 \, f^j = -\int_{\partial V} da \, n_i T^{ij}}$$



Problem



- Calculate the force per area on the interface. Method I; energy. Method 2 stress tensor

$$F^z = -\int_{\text{dased line}} dS \; n_i T^{ij}$$

$$= -A T_2^{zz} + A T_1^{zz}$$

Now

$$-T_2^{zz} = E^z D^z - \frac{1}{2} E \cdot D \, \delta^{zz} = +\frac{1}{2} E^z D^z = \frac{1}{2} \frac{(D_2^z)^2}{\varepsilon_2}$$

and similarly, $T_1^{zz} = \frac{1}{2} \frac{(D_1^z)^2}{\varepsilon_1}$. So

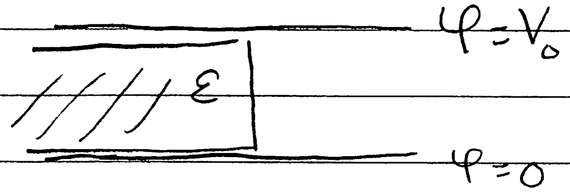
$$F^z = \frac{1}{2} \frac{(D_2^z)^2}{\varepsilon_2} - \frac{1}{2} \frac{(D_1^z)^2}{\varepsilon_1}$$

Using continuity of D at interface, $D_2 - D_1 = \cancel{\sigma_f}$. An[d] at the metal plates $\vec{n} \cdot \vec{D} = \sigma_f$. Yielding

$$\boxed{F^z = \frac{1}{2} \sigma_f^2 \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1} \right)}$$

Problem

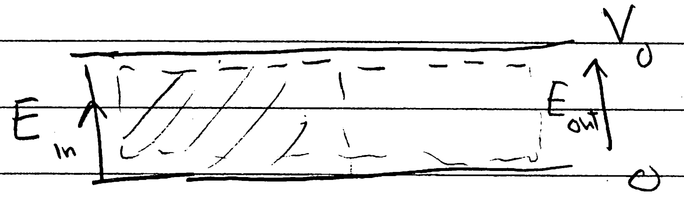
Consider the following parallel plate capacitor held



Use the stress tensor to determine the force on the dielectric

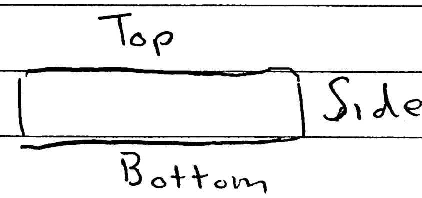
Solution

First note that the potential is constant so the electric field is the same at E_{in} and E_{out}. The charge density is not the same. In particular



$$\sigma_{in} = D_{in} = \varepsilon E_{in} = \varepsilon E_{out}, \quad \text{while} \quad \sigma_{out} = E_{out}$$

Then using the stress tensor on the dashed volume shown



$$F^x = -\oint da\, n_i T^{ix}$$

$$F^x = -\underset{\text{side}}{A}\left(T^{xx}_{out} - T^{xx}_{in}\right)$$

↖ area of side face

The stress tensor on the outside

$$T^{xx}_{out} = -E^x_{out}\cancelto{0}{E^x_{out}} + \frac{1}{2}E^2_{out}\,\delta^{xx}$$

$$T^{xx}_{out} = \frac{1}{2}E^2_{out}$$

while inside

$$T^{xx}_{in} = -\cancelto{0}{E^x_{in}}D^x_{in} + \frac{1}{2}E_{in}\cdot D_{in}\,\delta^{xx}$$

$$= \frac{1}{2}E_{in}^2\,\varepsilon$$

So the force per area

$$\frac{F^x}{A_{side}} = -\frac{1}{2}E^2_{out} + \frac{1}{2}\varepsilon E^2_{in} \qquad E_{in} = E_{out}$$

$$= \frac{1}{2}E^2_{out}\left(\varepsilon - 1\right)$$

$$\boxed{\frac{F^x}{A_{side}} = \frac{1}{2}\frac{V_0^2}{d^2}(\varepsilon - 1)}$$