

4 Electric Fields in Matter

4.1 Parity and Time Reversal

(a) We discussed how fields transform under parity and time reversal. A useful table is

Quantity	Parity	Time Reversal
t	Even	Odd
\mathbf{r}	Odd	Even
\mathbf{p}	Odd	Odd
\mathbf{F} =force	Odd	Even
$\mathbf{L} = \mathbf{r} \times \mathbf{p}$	Even	Odd
Q = charge	Even	Even
\mathbf{j}	Odd	Odd
\mathbf{E}	Odd	Even
\mathbf{B}	Even	Odd
\mathbf{A} vector potential	Odd	Odd

4.2 Electrostatics in Material

Basic setup

(a) In material we expand the medium currents \mathbf{j}_{mat} in terms of a constitutive relation, fixing the currents in terms of the applied fields.

$$\mathbf{j}_{mat} = [\text{all possible combinations of the fields and their derivatives}] \quad (4.1)$$

We have added a subscript *mat* to indicate that the current is a medium current. There is also an external current \mathbf{j}_{ext} and charge density ρ_{ext} .

(b) When only uniform electric fields are applied, and the electric field is weak, and the medium is isotropic, the polarization current takes the form

$$\mathbf{j}_{mat} = \sigma \mathbf{E} + \chi \partial_t \mathbf{E} + \dots \quad (4.2)$$

where the ellipses denote higher time derivatives of electric fields, which are suppressed by powers of t_{micro}/T_{macro} by dimensional analysis. For a conductor σ is non-zero. For a dielectric insulator σ is zero, and then the current takes the form

$$\mathbf{j}_b = \partial_t \mathbf{P} \quad (4.3)$$

- \mathbf{P} is known as the polarization, and can be interpreted as the dipole moment per volume.

- We have worked with linear response for an isotropic medium where

$$\mathbf{P} = \chi \mathbf{E} \quad (4.4)$$

This is most often what we will assume.

For an anisotropic medium, χ is replaced by a susceptibility tensor

$$P_i = \chi_{ij} E^j \quad (4.5)$$

For a nonlinear (isotropic) medium \mathbf{P} one could try a non-linear vector function of \mathbf{E} ,

$$\mathbf{P}(\mathbf{E}) \quad (4.6)$$

defined by the low-frequency expansion of the current at zero wavenumber.

- (c) Current conservation $\partial_t \rho + \nabla \cdot \mathbf{j} = 0$ determines then that

$$\rho_{mat} = -\nabla \cdot \mathbf{P} \quad (4.7)$$

- (d) The electrostatic maxwell equations read

$$\nabla \cdot \mathbf{E} = -\underbrace{\nabla \cdot \mathbf{P}}_{\rho_{mat}} + \rho_{ext} \quad (4.8)$$

$$\nabla \times \mathbf{E} = 0 \quad (4.9)$$

or

$$\nabla \cdot \mathbf{D} = \rho_{ext} \quad (4.10)$$

$$\nabla \times \mathbf{E} = 0 \quad (4.11)$$

where the *electric displacement* is

$$\mathbf{D} \equiv \mathbf{E} + \mathbf{P} \quad (4.12)$$

- (e) For a linear isotropic medium

$$\mathbf{D} = (1 + \chi) \mathbf{E} \equiv \varepsilon \mathbf{E} \quad (4.13)$$

but in general \mathbf{D} is a function of \mathbf{E} which must be specified before problems can be solved.

Working problems with Dielectrics

- (a) Using Eq. (4.7) and the Eq. (4.10) we find the boundary conditions that *normal* components of \mathbf{D} jump across a surface if there is external charge, while the *parallel* components \mathbf{E} are continuous

$$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_{ext} \quad D_{2\perp} - D_{1\perp} = \sigma_{ext} \quad (4.14)$$

$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \quad E_{2\parallel} - E_{1\parallel} = 0 \quad (4.15)$$

Very often σ_{ext} will be absent and then D_{\perp} will be continuous (but *not* E_{\perp}).

- (b) A jump in the polarization induces bound surface charge at the jump.

$$-\mathbf{n} \cdot (\mathbf{P}_2 - \mathbf{P}_1) = \sigma_{mat} \quad (4.16)$$

- (c) Since the curl of \mathbf{E} is zero we can always write

$$\mathbf{E} = -\nabla \varphi \quad (4.17)$$

and for linear media ($\mathbf{D}(\mathbf{r}) = \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})$) with a non-constant dielectric constant $\varepsilon(\mathbf{r})$, we find an equation for \mathbf{D}

$$\nabla \cdot \varepsilon(\mathbf{r}) \nabla \varphi = 0 \quad (4.18)$$

- (d) With the assumption of a linear medium $\mathbf{D} = \varepsilon \mathbf{E}$ and constant dielectric constant, the equations for electrostatics in medium are essentially identical to electrostatics without medium

$$-\varepsilon \nabla^2 \Phi = \rho_{ext}, \quad (4.19)$$

but, the new boundary conditions lead to some (pretty minor) differences in the way the problems are solved.

Energy and Stress in Dielectrics: Lecture 13.5

- (a) We worked out the extra energy stored in a dielectric as an ensemble of external charges are placed into the dielectric. As the macroscopic electric field \mathbf{E} and displacement $\mathbf{D}(\mathbf{E})$ are changed by adding external charge $\delta\rho_{ext}$, the change in energy stored in the capacitor material is

$$\delta U = \int_V d^3x \mathbf{E} \cdot \delta \mathbf{D} \quad (4.20)$$

- (b) For a linear dielectric δU can be integrated, becoming

$$U = \frac{1}{2} \int_V d^3x \mathbf{E} \cdot \mathbf{D} = \frac{1}{2} \int_V d^3x \varepsilon \mathbf{E}^2 \quad (4.21)$$

- (c) We worked out the stress tensor for a linear dielectric and found

$$T_E^{ij} = -\frac{1}{2}(D^i E^j + E^i D^j) + \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \delta^{ij} \quad (4.22)$$

$$= \varepsilon \left(-E^i E^j + \frac{1}{2} \mathbf{E}^2 \delta^{ij} \right) \quad (4.23)$$

where in the first line we have written the stress in a form that can generalize to the non-linear case, and in the second line we used the linearity to write it in a form which is proportional the vacuum stress tensor.

- (d) As always the force per volume in the Dielectric is

$$f^j = -\partial_i T_E^{ij} \quad (4.24)$$

where

$$T^{ij} = \text{the force in the } j\text{-th direction per area in the } i\text{-th} \quad (4.25)$$

More precisely let \mathbf{n} be the (outward directed) normal pointing from region LEFT to region RIGHT, then

$$n_i T^{ij} = \text{the } j\text{-th component of the force per area, by region LEFT on region RIGHT} \quad (4.26)$$

We can integrate the force/volume to find the net force on a given volume

$$F^j = \int_V d^3x f^j(\mathbf{x}) = - \int_{\partial V} da_i T^{ij} \quad (4.27)$$

This can be used to work out the force at a dielectric interface as done in lecture.