Retarded Green Functions

- Our goal is to write down the retarded Green function of the Maxwell equation and to learn mathematics.

- Let us start with the harmonic oscillator

\[
\left[ \frac{md^2}{dt^2} + m\gamma \frac{d}{dt} + mw^2 \right] G_p(t,t_0) = S(t-t_0)
\]

\[
\equiv \mathcal{L}_t
\]

\(G_p(t,t_0)\) is the displacement at time \(t\), due to an impulsive force at time \(t_0\). Here we have defined the linear operator \(\mathcal{L}_t\) which is the underlined term. For a general \(F(t)\) driving the oscillator

\[
\mathcal{L}_t x(t) = F(t)
\]

The general solution is a specific solution \(X_s(t)\) (usually the steady state) + a homogeneous solution \(X_{\text{homo}}(t)\)

\[
x(t) = X_s(t) + X_{\text{homo}}(t)
\]

Where

\[
\mathcal{L}_t X_s(t) = F(t) \quad \text{and} \quad \mathcal{L}_t X_{\text{homo}}(t) = 0
\]

and \(X_{\text{homo}}\) is adjusted to satisfy the initial conditions
For the oscillator example at small damping

\[ X(t) = A e^{-\frac{1}{2} t} e^{-i\omega t} + B e^{-\frac{1}{2} t} e^{i\omega t} \]

The specific solution

(1) \[ X_s(t) = \int_{-\infty}^{\infty} G(t-t_0) F(t_0) \, dt_0 \]

The homogeneous solution will decay away in time leaving the specific solution. This clearly satisfies the equation

\[ \mathcal{L} X_s(t) = \int_{-\infty}^{\infty} \mathcal{L} G(t-t_0) F(t_0) \, dt_0 \]

\[ = \int_{-\infty}^{\infty} S(t-t_0) F(t_0) \, dt_0 = F(t) \]

We will specifically be interested in the retarded or causal Green function:

\[ G_R(t,t_0) = 0 \quad \text{for} \quad t < t_0 \]

So \( G_R \) is a response at \( t \) to a force at \( t_0 \).
Note all physical quantities are ultimately expressible as Grn-fans. For example, we used a harmonic oscillator (Lorentz Model) to describe the dielectric constant, \( \tilde{F}(t) = eElt \) the current \( \tilde{j}(t) = NEV(t) \), so from Eq. (1)

\[
X_e(\omega) = G_R(\omega) F(\omega)
\]

And

\[
\tilde{j}(\omega) = ne \left( -i\omega x(\omega) \right)
\]

\[
= ne^2 G_R(\omega) \left( -i\omega E(\omega) \right)
\]

So comparison with the constitutive relation \( j(\omega) = x_e(\omega) (-i\omega E(\omega)) \) gives

\[
x_e(\omega) = ne^2 G_R(\omega)
\]

Thus we see how in a particular model, the response function of the dynamical system determines the susceptibility.
Finding the Green Function in time: Direct Method

\[
\left[ \frac{md^2}{dt^2} + m \eta \frac{d}{dt} + mw_0^2 \right] G_R(t, t_0) = \delta(t-t_0)
\]

Demand continuity and integrate from \( t_0 - \epsilon \) to \( t_0 + \epsilon \). We know \( G_R(t, t_0) = 0 \) for \( t < t_0 \)

\[ G_R(t + \epsilon, t_0) = 0 \]

Then we have

\[
m \frac{d}{dt} G_R(t + \epsilon, t_0) + m \eta G_R(t + \epsilon, t_0) = 1
\]

\[ \lim_{\epsilon \to 0} \]

Or

\[ m \frac{d}{dt} G_R(t + \epsilon, t_0) = 1 \]

Then we can solve the diff-eq given the initial conditions. The two homogeneous solutions are

\[ x_\pm = e^{-\gamma/2} e^{+i \omega_0 t} \text{ for small } \eta \]

Then the linear combo of \( x_\pm \) which satisfies the initial conditions (\( \star \)) and (\( \star \star \)) are

\[
G_R = \begin{cases} 
\sin \omega_0 (t-t_0) e^{-\gamma/2 (t-t_0)}/mw_0 & t-t_0 > 0 \\
0 & \text{otherwise}
\end{cases}
\]
Usually this is written

\[ G_e(t) = \Theta(t) \sin \omega_0 t \frac{e^{-\frac{\pi}{2}}}{{\pi} \frac{t}{m\omega_0}} \quad t \equiv t-t_0 \]
Fourier Method for Green function

\[ \int \frac{e^{-i\omega \tau}}{2\pi} \]

\[ \left[ \frac{m}{\alpha^2} \frac{d^2}{d\tau^2} + m\eta \frac{d}{d\tau} + m\omega_0^2 \right] G_R(\tau) = \delta(\tau) \]

Fourier Transform both sides

\[ \left[ -m\omega^2 + m\eta(-i\omega) + m\omega_0^2 \right] G_R(\omega) = 1 \]

\[ G_R(\omega) = \frac{1/m}{\left[ -\omega^2 + \omega_0^2 - i\omega\eta \right]} \]

Thus

\[ G_R(\tau) = \int \frac{e^{-i\omega \tau}}{2\pi} \frac{1/m}{\left[ -\omega^2 + \omega_0^2 - i\omega\eta \right]} \]

You can do these integrals with contour integration.

the poles are at

\[ \omega^2 + i\omega\eta = \omega_0^2 \]

\[ \omega \sim \pm \omega_0 - i\eta \]

Solving this equation for small \( \eta \):

We see that the integrand has the following analytic structure.
So now we should do the integral:

Case 1: $T < 0 \quad G_R(t) = 0 \iff \text{causality}$

The math works like this, since $T < 0$:

$e^{-i\omega T} \xrightarrow{\text{complex}} e^{i\Re(\omega)T} e^{-[\Im(\omega)T]}$

decreasing exponentially for $\Im(\omega) > 0$

Thus for $T < 0$ we can close the contour in the UHP without picking up poles and find zero
Case 2: \( t > 0 \)

For \( t > 0 \) we must close the contour in the LHP picking up poles at \( \omega = \pm \omega_0 - \frac{i\gamma}{2} \).

For \( t > 0 \): wrong way around poles

\[
G_R(t) = -2\pi i \left[ \text{Res}_{\omega = \omega_0 - \frac{i\gamma}{2}} + \text{Res}_{\omega = -\omega_0 - \frac{i\gamma}{2}} \right]
\]

\[
= \frac{1 - i}{m 2\omega_0} e^{-\frac{\gamma}{2} t} e^{-i\omega_0 t} + \frac{1 + i}{m 2\omega_0} e^{-\frac{\gamma}{2} t} e^{+i\omega_0 t}
\]

\[
= \frac{1}{m 2\omega_0} e^{-\frac{\gamma}{2} t} \sin \omega_0 t
\]

So

\[
G_R(t) = \Theta(t) \sin \omega_0 t e^{-\frac{\gamma}{2} t} \xrightarrow{\gamma \to 0} \Theta(t) \sin \frac{\omega_0 t}{m \omega_0}
\]

We will see that this Green function is closely related to the Green function of the wave eqn.
(Aside: i.e. prescription:)

Take the $\gamma \to 0$ limit of the damped harmonic oscillator

$$G_p(t) = \frac{\sin \omega_0 T}{m \omega_0} \Theta(t)$$

$$G_p(\omega) = \frac{Vm}{[-\omega^2 + \omega_0^2]}$$

But this is ambiguous since the poles are on the real axis. What does this mean $\int \frac{dw}{2\pi i} \frac{e^{-i\omega t}}{[-\omega^2 + \omega_0^2]}$?

We know that causality demands that the poles lie in the lower half plane. We can enforce this by adding an infinitesimal imaginary part

$$\omega \to \omega + i\epsilon$$ (positive)

So

$$G_p(\omega) = \frac{Vm}{-((\omega + i\epsilon)^2 + \omega_0^2)}$$

Amounts to adding an infinitesimal damping coefficient $\gamma = 2\epsilon$
Kramers–Krönig and retarded Green functions

The Kramers–Krönig relation holds for causal response functions, which are always analytic in UHP (upper half plane). \( G_R(w) \) satisfies these properties, thus:

\[
\text{Re} \ G_R(w) = - \int_{-\infty}^{\infty} \frac{dw'}{\pi} \frac{P \text{Im} \ G_R(w')}{w-w'}
\]

\[
\text{Im} \ G_R(w) = + \int_{-\infty}^{\infty} \frac{dw'}{\pi} \frac{P \text{Re} \ G_R(w')}{w-w'}
\]