Problem 1. Units

(a) Show that electric field and magnetic field have units $\sqrt{(\text{force})/\text{area}}$ or $\sqrt{\text{energy}/\text{volume}}$.

(b) A rule of thumb that you may need in the lab is that coaxial cable has a capcitance of 12 pF/foot. That is why cable length must be kept to a minimum in high speed electronics.

The order of magnitude of this result is set by $\epsilon_o = 8.85\,\text{pF/m}$. In the Heavyside-Lorentz system capacitance is still $Q_{HL} = C_{HL} V_{HL}$. Show that C_{HL} has units of meters, and that

$$C_{MKS} = 8.85\, pF \left(\frac{C_{HL}}{\text{meters}} \right) \tag{1}$$

(c) The "impedance of the vacuum" is $Z_o = \sqrt{\mu_o/\epsilon_o} = 376\,\text{Ohms}$. This is why high frequency antennas will typically have a "radiation resistance" of this order of magnitude. As this problem will discuss, the unit of resistance is s/m for the Heavyside Lorentz system, and "the impedance of the vacuum" is $1/c$

In Heavyside-Lorentz units Ohm's law still reads, $\boldsymbol{j}_{HL} = \sigma_{HL} \boldsymbol{E}_{HL}$, where σ_{HL} is the conductivity, and $\boldsymbol{j}$ is the current per area. Show that the conductivity in Heavyside-Lorentz has units $[\sigma_{HL}] = 1/\text{seconds}$ and that $\sigma_{MKS} = \sigma_{HL}\epsilon_o$. Then show that a wire of length L and radius R_o has resistance

$$R_{MKS} = 376\,\text{Ohms}\,(R_{HL} c) \tag{2}$$

$$= 376\,\text{Ohms} \left(\frac{Lc}{\pi R_o^2 \sigma_{HL}} \right) \tag{3}$$

What is σ_{HL} for copper?

Problem 2. Vector Identities

(a) Use the epsilon tensor to prove the analog of "b(ac)-(ab)c" rule for curls

$$\nabla \times (\nabla \times \boldsymbol{V}) = \nabla(\nabla \cdot \boldsymbol{V}) - \nabla^2 \boldsymbol{V} \tag{4}$$

Use this result, together with the Maxwell equations in the absence of charges and currents, to establish that $\boldsymbol{E}$ and $\boldsymbol{B}$ obey the wave equation

$$\frac{1}{c^2}\partial_t^2 \boldsymbol{B} - \nabla^2 \boldsymbol{B} = 0 \tag{5}$$

$$\frac{1}{c^2}\partial_t^2 \boldsymbol{E} - \nabla^2 \boldsymbol{E} = 0 \tag{6}$$

(b) When differentating $1/r$ we write

$$\frac{1}{r} = \frac{1}{\sqrt{x^i x_i}} \tag{7}$$

with $\boldsymbol{x} = x^i \boldsymbol{e}_i$, and use results like

$$\partial_i x^j = \delta_i^j \qquad \partial_i x^i = \delta_i^i = d = 3 \tag{8}$$

where $d = 3$ is the number of spatial dimensions. (It is usually helps to write this as d rather than 3 to get the algebra right). In this way, one finds that field due to a electric charge (monopole) is the familiar $\hat{\boldsymbol{r}}/r^2$

$$j\text{-th component of} \quad -\nabla(1/r) = \left(-\nabla\frac{1}{r}\right)_j = -\partial_j \frac{1}{\sqrt{x^i x_i}} = \frac{\frac{1}{2}(x^i \delta_{ji} + x_i \delta_j^i)}{(x^k x_k)^{3/2}} = \frac{x_j}{r^3} = \frac{(\hat{\boldsymbol{r}})_j}{r^2} \tag{9}$$

where $\hat{\boldsymbol{r}} \equiv \boldsymbol{n} = \boldsymbol{x}/r$.

Using tensor notation (*i.e.* indexed notation) show that

$$\nabla \times \frac{\hat{\boldsymbol{r}}}{r^2} = 0 \tag{10}$$

(c) Using the tensor notation (*i.e.* indexed notation) show that for constant vector $\boldsymbol{p}$ (and away from $\boldsymbol{r} = 0$) that

$$-\nabla\left(\frac{\boldsymbol{p}\cdot\boldsymbol{n}}{4\pi r^2}\right) = \frac{3(\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n} - \boldsymbol{p}}{4\pi r^3} \tag{11}$$

Remark: $\phi_{\rm dip} = \boldsymbol{p}\cdot\boldsymbol{n}/(4\pi r^2)$ is the electrostatic potential due to an electric dipole $\boldsymbol{p}$, and Eq. (11) records the corresponding electric field. Notice the $1/r^3$ as opposed to $1/r^2$ for the monopole, and, taking $\boldsymbol{p}$ along the z-axis, notice how the electric field points at $\theta = 0$ (or $\boldsymbol{n} = \hat{\boldsymbol{z}}$) and $\theta = \pi/2$ (or $\boldsymbol{n} = \hat{\boldsymbol{x}}$). How could you derive this using the identities on the front cover of Jackson?

Problem 3. Easy important application of Helmholtz theorems

(a) Using the source free Maxwell equations (*i.e.* those without ρ and $\boldsymbol{j}$) and the Helmholtz theorems, explain why $\boldsymbol{E}$ and $\boldsymbol{B}$ can be written in terms of a scalar field Φ (the scalar potential) and a vector field $\boldsymbol{A}$ (the vector potential)

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{12}$$

$$\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\Phi \tag{13}$$

Thus two of the four Maxwell equations are trivially solved by introducing Φ and $\boldsymbol{A}$.

(b) Show that $\boldsymbol{A}$ and Φ are not unique, *i.e.*

$$A_i = (A_{\text{old}})_i + \partial_i \Lambda(t, \boldsymbol{x}) \tag{14}$$

$$\Phi = (\Phi_{\text{old}}) - \frac{1}{c}\partial_t \Lambda(t, \boldsymbol{x}) \tag{15}$$

gives the same $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. Here $\Lambda(t, \boldsymbol{x})$ is any function. This change of fields is known as a gauge transformation of the gauge fields (Φ, $\boldsymbol{A}$).

(c) Now, using the sourced Maxwell equations (*i.e.* those with ρ and $\boldsymbol{j}$), show that current must obey the conservation Law

$$\partial_t \rho + \nabla \cdot \boldsymbol{j} = 0\,, \tag{16}$$

to be consistent with the Maxwell equations.

Problem 4. Tensor decomposition

(a) Consider a tensor T^{ij}, and define the symmetric and anti-symmetric components

$$T_S^{ij} = \tfrac{1}{2}\left(T^{ij} + T^{ji}\right) \tag{17}$$

$$T_A^{ij} = \tfrac{1}{2}\left(T^{ij} - T^{ji}\right) \tag{18}$$

so that $T^{ij} = T_S^{ij} + T_A^{ij}$. Show that the symmetric and anti-symmetric components don't mix under rotation

$$\underline{T_S}^{ij} = R^i{}_\ell R^j{}_m T_S^{\ell m} \tag{19}$$

$$\underline{T_A}^{ij} = R^i{}_\ell R^j{}_m T_A^{\ell m} \tag{20}$$

This means that I don't need to know T_A if I want to find $\underline{T_S}$ in a rotated coordinate system.

Remarks: We say that the general rank two tensor is reducable to $T^{ij} = T_S^{ij} + T_A^{ij}$ into two tensors that dont mix under rotation

(b) You should recognize that an antisymmetric tensor is isomorphic to a vector

$$V_i \equiv \tfrac{1}{2}\epsilon_{ijk} T_A^{jk} \tag{21}$$

Explain qualitatively the identity $\epsilon^{ijk}\epsilon_{\ell mk} = \delta^i{}_\ell \delta^j{}_m - \delta^j{}_\ell \delta^i{}_m$ using $\epsilon^{ij3}\epsilon_{\ell m3}$ as an example, and use this to show

$$T_A^{ij} = \epsilon^{ijk} V_k \tag{22}$$

Remark: In matrix form this reads

$$T_A = \begin{pmatrix} 0 & V_z & -V_y \\ -V_z & 0 & V_x \\ V_y & -V_x & 0 \end{pmatrix} \tag{23}$$

(c) Using the Einstein summation convention, show that the trace of a symmetric tensor is rotationally invariant

$$\underline{T}^i{}_i \equiv T^i{}_i \tag{24}$$

and that

$$\mathring{T}_S^{ij} \equiv T^{ij} - \tfrac{1}{3}\delta^{ij} T^\ell{}_\ell \tag{25}$$

is traceless.

Remark: A symmetric tensor is therefore reducable to a symmetric traceless tensor and a scalar times δ^{ij}.

$$T_S^{ij} = \mathring{T}_S^{ij} + \tfrac{1}{3}\delta^{ij} T^\ell{}_\ell \qquad \text{where} \qquad \mathring{T}_S^{ij} \equiv T_S^{ij} - \tfrac{1}{3} T^\ell{}_\ell \delta^{ij} \tag{26}$$

I don't need to know T^ℓ_ℓ in order to compute $\underline{\mathring{T}_S^{ij}} = R^i{}_\ell R^j{}_m \mathring{T}_S^{\ell m}$

Remarks: The results of this problem show that a general second rank tensor is decomposable into irreducable components

$$T^{ij} = \mathring{T}_S^{ij} + \epsilon^{ijk} V_k + \tfrac{1}{3} T^\ell_\ell \delta^{ij} \tag{27}$$

$$= \frac{1}{2}\left(T^{ij} + T^{ji} - \tfrac{2}{3} T^\ell_\ell \delta^{ij}\right) + \frac{1}{2}\epsilon^{ijk}\epsilon_{k\ell m} T^{\ell m} + \frac{1}{3} T^\ell_\ell \delta^{ij} \tag{28}$$

No further reduction is possible. A general result is that a fully symmetric traceless tensor is irreducable.

When this result is applied to the product of two vectors it says

$$E^i B^j = \frac{1}{2}\left(E^i B^j + B^i E^j - \tfrac{2}{3} \boldsymbol{E}\cdot\boldsymbol{B}\delta^{ij}\right) + \frac{1}{2}\epsilon^{ijk}(\boldsymbol{E}\times\boldsymbol{B})_k + \frac{1}{3}\boldsymbol{E}\cdot\boldsymbol{B}\delta^{ij} \tag{29}$$

which expresses the tensor product of two vectors as the sum of an irreducable (traceless and symmetric) tensor, a vector, and a scalar, $1 \otimes 1 = 2 \oplus 1 \oplus 0$.

More physically it says that not all of $E_i B_j$ is really described by a tensor. Rather, part of $E_i B_j$ is described by the vector $\boldsymbol{E}\times\boldsymbol{B}$, and part is described by the scalar $\boldsymbol{E}\cdot\boldsymbol{B}$. It is for this reason that the tensors we work with in physics (*i.e.* the moment of inertia tensor, the quadrupole tensor, the maxwell stress tensor) are symmetric and traceless.

Problem 5. 3d delta-functions

A delta-function in 3 dimensions $\delta^3(\boldsymbol{r} - \boldsymbol{r}_o)$ is an infinitely narrow spike at $\boldsymbol{r}_o$ which satisfies

$$\int d^3\boldsymbol{r}\, \delta^3(\boldsymbol{r} - \boldsymbol{r}_o) = 1 \tag{30}$$

In spherical coordinates, where the measure is

$$d^3\boldsymbol{r} = r^2 dr\, d(\cos\theta)\, d\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi\,, \tag{31}$$

we must have

$$\delta^3(\boldsymbol{r} - \boldsymbol{r}_o) = \frac{1}{r^2}\delta(r - r_o)\delta(\cos\theta - \cos\theta_o)\delta(\phi - \phi_o) = \frac{1}{r^2 \sin\theta}\delta(r - r_o)\;\delta(\theta - \theta_o)\delta(\phi - \phi_o) \tag{32}$$

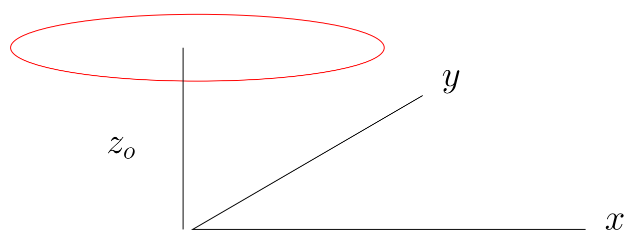
so that $\int d^3\boldsymbol{r}\, \delta^3(\boldsymbol{r}) = 1$. For a general curvlinear coordinate system

$$\delta^3(\boldsymbol{r} - \boldsymbol{r}_o) = \frac{1}{\sqrt{g}} \prod_a \delta(u^a - u^a_o) \tag{33}$$

where u^a_o are the coordinates of $\boldsymbol{r}_o$.

(a) What is formula $\delta^3(\boldsymbol{r} - \boldsymbol{r}_o)$ for cylindrical coordinates?

(b) A uniform ring of charge Q and radius a sits at height z_o above the xy plane, and the plane of the ring is parallel to the xy plane. Express the charge density $\rho(\boldsymbol{r})$ (charge per volume) in spherical coordinates using delta-functions. Check that the volume integral of $\rho(\boldsymbol{r})$ gives the total Q.



Problem 6. Fourier Transforms of the Coulomb Potential

The fourier transfrom takes a function in coordinate space and represents in momentum space[1]

$$F(k) = \int_{-\infty}^{\infty} dx \left[e^{-ikx}\right] f(x) \tag{34}$$

The inverse transformation repesents a function as a sum of plane waves

$$F(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[e^{ikx}\right] F(k) \tag{35}$$

The Fourier transform generalizes the concept of a fourier series to non-periodic, but square integrable functions – *i.e.* $\int dx |f(x)|^2$ should converge.

The Fourier transform of a 3D function $\boldsymbol{r} = (x, y, z)$ is:

$$F(\boldsymbol{k}) = \int d^3\boldsymbol{r} \, \left[e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right] F(\boldsymbol{r}) \tag{36}$$

$$F(\boldsymbol{r}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \, \left[e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right] F(\boldsymbol{k}) \tag{37}$$

To do this problem you will need to know (as dicussed in class) that the integral of a pure phase e^{ikx} is proportional to a delta-fcn. In $3D$ we have

$$\delta^3(\boldsymbol{r}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{38}$$

$$(2\pi)^3 \delta^3(\boldsymbol{k}) = \int d^3\boldsymbol{r} \, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{39}$$

I find it useful to abbreviate these integrals (try it!)

$$\int_{\boldsymbol{k}} \equiv \int \frac{d^3k}{(2\pi)^3} \qquad \int_{\boldsymbol{r}} \equiv \int d^3\boldsymbol{r} \tag{40}$$

Thus we have

$$\int_{\boldsymbol{k}} \int_{\boldsymbol{r}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} = 1 \tag{41}$$

(a) Use tensors notation to show that the Fourier transform of $\nabla F(\boldsymbol{r})$ is

$$i\boldsymbol{k} F(\boldsymbol{k}) \,, \tag{42}$$

and that the Fourier transform of the curl of a vector vector field $\boldsymbol{F}(\boldsymbol{r})$ is $\nabla \times \boldsymbol{F}(\boldsymbol{r})$ is

$$i\boldsymbol{k} \times \boldsymbol{F}(\boldsymbol{k}) \tag{43}$$

(b) The genral rule is to replace $\nabla \to i\boldsymbol{k}$. What is the Fourier transform of $\nabla^2 F(\boldsymbol{r})$

[1] The notation of putting e^{ikx} in square brackets is not standard, but I have used it in the notes to highlight the similarity between this expansion and other eigenfunction expansions.

(c) Prove the Convolution Theorem, *i.e.* the Fourier Transform of a product is a convolution

$$\int d^3\boldsymbol{r}\, e^{-i\Delta\boldsymbol{k}\cdot\boldsymbol{r}}\, |F(\boldsymbol{r})|^2 = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} F(\boldsymbol{k})F^*(\boldsymbol{k}-\Delta\boldsymbol{k}) \tag{44}$$

making liberal use of the completeness integrals

$$\int d^3\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} = (2\pi)^3\delta^3(\boldsymbol{k}) \tag{45}$$

Remark: Setting $\Delta\boldsymbol{k} = 0$ we recover Parseval's Theorem

$$\int d^3r |F(\boldsymbol{r})|^2 = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} |F(\boldsymbol{k})|^2 \tag{46}$$

Remark: This is often used in reverse, the fourier transform of a convolution is a product of the fourier transforms

$$\text{F.T. of } \int d^3\boldsymbol{r}_o\, F(\boldsymbol{r}_o)\, G(\boldsymbol{r}-\boldsymbol{r}_o) \; = F(\boldsymbol{k})G(\boldsymbol{k}) \tag{47}$$

(d) The Fourier transform of the Coulomb potential is difficult (try it and find out why!). This is because $1/(4\pi r)$ is not in the space of square integrable functions (Why?). Thus, we will consider the Fourier transform of $1/(4\pi r)$ to be the limit as $m \to 0$ of the Fourier transform of a screened Coulomb potential known as the Yukawa potential

$$\Phi(\boldsymbol{x}) = \frac{e^{-m|\boldsymbol{r}|}}{4\pi|\boldsymbol{r}|} \tag{48}$$

The Yukawa potential is square integrable. Show that the Fourier transform of the Yukawa potential is

$$\Phi(\boldsymbol{k}) = \frac{1}{k^2+m^2} \tag{49}$$

with $k = \sqrt{\boldsymbol{k}^2}$. Thus, we conclude with $m \to 0$ that

$$\int d^3\boldsymbol{x}\; e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, \frac{1}{4\pi r} = \frac{1}{k^2} \tag{50}$$

Note that the inverse transform can be computed by direct integration

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|} = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_o)}}{k^2} \tag{51}$$

(e) In electrostatics the electric field is the negative gradient of the potential, $\boldsymbol{E} = -\nabla\Phi$. From $\nabla\cdot\boldsymbol{E} = \rho$, we derive the Poisson equation $-\nabla^2\Phi = \rho$. For a unit charge at the origin, the coulomb potential, $1/(4\pi r)$, satisfies

$$-\nabla^2\Phi = \delta^3(\boldsymbol{r}) \tag{52}$$

Deduce Eq. (50) by fourier transforming this equation.