Problem 1. A cylinder in a magnetic field

A very long hollow cylinder of inner radius a and outer radius b of permeability μ is placed in an initially uniform magnetic field $\boldsymbol{B}_o$ at right angles to the field.

(a) For a constant field B_o in the x direction show that $A^z = B_o y$ is the vector potential. This should give you an idea of a convenient set of coordinates to use.

Remark: See Wikipedia for a list of the vector Laplacian in all coordinates. Most often the vector Laplacian is used if the current is azimuthal and solutions may be looked for with $A_\phi \neq 0$ and $A_r = A_\theta = 0$ (or $A_\rho = A_z = 0$ in cylindrical coordinates). This could be used for example in Problem 3. Similarly if the current runs up and down, with $A_z \neq 0$ and $A_\rho = A_\phi = 0$, so that $\boldsymbol{B} = (B_x(x,y,z), B_y(x,y,z), 0)$ is independent of z, then the vector Laplacian in cylindrical coordinates $-\nabla^2 A_z$ is a good way to go.

(b) Show that the magnetic field in the cylinder is constant $\rho < a$ and determine its magnitude.

(c) Sketch $|\boldsymbol{B}|/|\boldsymbol{B}_o|$ at the center of the as function of μ for $a^2/b^2 = 0.9, 0.5, 0.1$ for $\mu > 1$.

Problem 2. Helmholtz coils

Consider a compact circular coil of radius a carrying current I, which lies in the $x-y$ plane with its center at the origin.

(a) By elementary means compute the magnetic field along the z axis.

(b) Show by direct analysis of the Maxwell equations $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{B} = 0$ that slightly off axis near $z = 0$ the magnetic field takes the form

$$B_z \simeq \sigma_0 + \sigma_2 \left(z^2 - \tfrac{1}{2}\rho^2\right), \quad B_\rho \simeq -\sigma_2 z \rho \,, \tag{1}$$

where $\sigma_0 = (B_z^o)$ and $\sigma_2 = \frac{1}{2}\left(\frac{\partial^2 B_z^o}{\partial z^2}\right)$ are the field and its z derivatives evaluated at the origin. For later use give σ_0 and σ_2 explictly in terms of the current and the radius of the loop.

Remark: Upon solving this problem, it should be clear that this method of solution does not rely on being close to $z = 0$. We just chose $z = 0$ for definiteness.

(c) Now consider a second identical coil (co-axial with the first), having the same magnitude and direction of the current, at a height b above the first coil, where a is the radii of the rings. With the coordinate origin relocated at the point midway between the two centers of the coils, determine the magnetic field on the z-axis near the origin as an expansion in powers of z to z^4. Use mathematica if you like. You should find that the coefficient of z^2 vanishes when $b = a$

Remark For $b = a$ the coils are known as Helmholtz coils. For this choice of b the z^2 terms in part (c) are absent. (Also if the off-axis fields are computed along the lines of part (b), they also vanish.) The field near the origin is then constant to 0.1% for $z < 0.17\,a$.

Problem 3. The field from a ring current.

Consider conducting ring of current radius a lying in the $x-y$ plane, carrying current I in the counter clockwise direction, $\boldsymbol{I} = I\hat{\boldsymbol{\phi}}$.

(a) Starting from the general (coulomb gauge) expression

$$\boldsymbol{A}(\boldsymbol{r}) = \int d^3\boldsymbol{r}_o \, \frac{\mathbf{j}(\boldsymbol{r}_o)/c}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|} \tag{2}$$

and the expansion of $1/(4\pi|\boldsymbol{r}-\boldsymbol{r}_o|)$ in spherical coordinates, show that the expansion of A_ϕ in the x, y plane inside the ring is

$$A_\phi(\rho)|_{z=0} = \frac{I}{2c}\sum_{\ell=1}^{\infty} \frac{(P^1_\ell(0))^2}{\ell(\ell+1)} \left(\frac{\rho}{a}\right)^\ell \tag{3}$$

where $\rho = \sqrt{x^2+y^2}$ and P^1_ℓ is the associated Legendre polynomial. (Check out wikipedia entry on spherical harmonics)

(b) Compute $B_z(\rho)$ in the x, y plane.

(c) Show that close to the axis of the shell the magnetic field you computed in part (b) is in agreement with the results of Eq. (1) when evaluated at $z = 0$, *i.e.* that for small ρ part (b) yields $B_z(\rho) \simeq \sigma_0 - \frac{1}{2}\sigma_2\rho^2$ with the appropriate values of σ_0 and σ_2.

Remark: Using the generating function of Legendre polynomials derived in class

$$\frac{1}{\sqrt{1+r^2-2r\cos\theta}} = \sum_{\ell=0}^{\infty} r^\ell P_\ell(\cos\theta) \tag{4}$$

and the defintion of $P^1_\ell(\cos\theta) = -\sin\theta \frac{dP_\ell(\cos\theta)}{d(\cos\theta)}$, we show that

$$\sum_{\ell=1}^{\infty} r^\ell P^1_\ell(0) = \frac{-r}{(1+r^2)^{3/2}} \simeq -r + \frac{3}{2}r^3 - \frac{15}{8}r^5 + \ldots \tag{5}$$

establishing that

$$P^1_1(0) = -1 \qquad P^1_3(0) = \frac{3}{2} \qquad P^1_5(0) = -\frac{15}{8} \qquad P^1_\ell(0) = 0 \text{ for } \ell \text{ even.} \tag{6}$$

(d) Consider a magnetic dipole of magnetic moment $\boldsymbol{m} = -m\hat{\boldsymbol{z}}$ in the $x-y$ plane oriented oppositely to the field from the ring, show that when the dipole is inside the ring the force on the dipole is

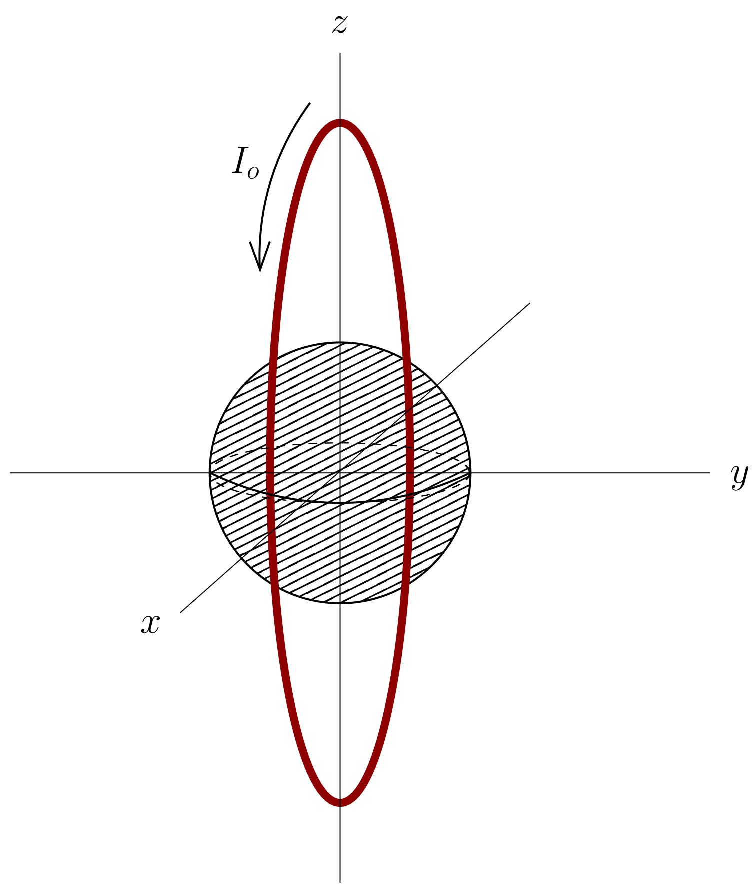
$$\mathbf{F} = -\hat{\boldsymbol{\rho}}\frac{mB_o}{a}\sum_{\ell=3}^{\infty} \frac{(\ell-1)}{\ell}(P^1_\ell(0))^2 \left(\frac{\rho}{a}\right)^{\ell-2} \tag{7}$$

where the negative indicates that the force is towards the center, and $B_o = I/(2ca)$ is the magnetic field in the center of the ring.

(e) Plot the force $|\mathbf{F}|\,/\,[mB_o/a]$ as a function of ρ/a.

Problem 4. A magnetized sphere and a circular hoop

A uniformly magnetized sphere of radius a centered at origin has a permanent total magnetic moment $\mathbf{m} = m\,\hat{\boldsymbol{z}}$ pointed along the z-axis (see below). A circular hoop of wire of radius b lies in the xz plane and is also centered at the origin. The hoop circles the sphere as shown below, and carries a small current I_o (which does not appreciably change the magnetic field). The direction of the current I_o is indicated in the figure.



(a) Determine the bound surface current on the surface of the sphere, and explain

(b) Determine the magnetic field $\boldsymbol{B}$ inside and outside the magnetized sphere by analogy with the spinning charged sphere disucssed in class.

(c) Show that your solution satisfies the boundary conditions of magnetostatics on the surface of the sphere.

(d) Compute the net-torque on the circular hoop. Indicate the direction and interpret.

Problem 5. Tensor reduction – easy and somewhat useful

In the following questions $\boldsymbol{x} = a(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a vector of length a. $\boldsymbol{v}$ is a vector with magnitude less than one. $f(\boldsymbol{x}\cdot\boldsymbol{v}) \equiv 1/(1+\boldsymbol{x}\cdot\boldsymbol{v})$ for definiteness.

Show the following:

(a)

$$\int d\Omega\, x^i x^j = \frac{4\pi a^2}{3}\delta^{ij} \tag{8}$$

(b)

$$\int d\Omega\, x^i f(\boldsymbol{x}\cdot\boldsymbol{v}) = \hat{v}^i I(v) \tag{9}$$

where $I(v) = \int d\Omega\, a\cos\theta/(1+v\cos\theta)$

(c) This will come up later in the course

$$\int d\Omega\, x^i x^j x^k x^l = \frac{4\pi a^4}{15}\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right) \tag{10}$$

(d) Show that

$$\int d\Omega\, x^i x^j f(\boldsymbol{x}\cdot\boldsymbol{v}) = C_1(v)\delta^{ij} + C_2(v)\left(\hat{v}^i\hat{v}^j - \frac{1}{3}\delta^{ij}\right) \tag{11}$$

where

$$C_1(v) = \frac{a^2}{3}\int d\Omega f(v\cos\theta) \tag{12}$$

$$C_2(v) = a^2\int d\Omega\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) f(v\cos\theta) \tag{13}$$

$$\tag{14}$$