Maxwell Equations + Induction + Energy in Mag fields

\[ \nabla \cdot E = \rho_{\text{mat}} + \rho_{\text{ext}} \]

\[ \nabla \times B = \frac{j_{\text{mat}} + j_{\text{ext}}}{c} + \frac{1}{c} \frac{\partial E}{\partial t} \]

\[ \nabla \cdot B = 0 \]

\[ -\nabla \times E = \frac{1}{c} \frac{\partial B}{\partial t} \]

Then for the material current we write

\[ j_{\text{mat}} = \frac{\sigma}{c} \frac{\partial}{\partial t} \vec{E} + \frac{1}{c} \frac{\partial}{\partial t} \vec{P} + \nabla \times \vec{M} \]

Then with continuity, \( \rho_{\text{mat}} = -\nabla \cdot \vec{P} \), \( \vec{D} = \vec{E} + \vec{P} \), \( \vec{H} = \vec{B} - \vec{M} \) find

\[ \nabla \cdot \vec{D} = \rho_{\text{ext}} \]

\[ \nabla \times \vec{H} = \frac{j_{\text{ext}}}{c} + \frac{1}{c} \frac{\partial}{\partial t} \vec{D} \]

\[ \nabla \cdot \vec{B} = 0 \]

\[ -\nabla \times \vec{E} = \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \]
Then we expand in powers of $c$.

**Electrostatics**

\[ \nabla \cdot D^{(0)} = \rho_{\text{ext}} \]

\[ \nabla \times E^{(0)} = 0 \]

**Magnetostatics**

\[ \nabla \times H^{(1)} = \frac{\mathbf{j}_{\text{ext}}}{c} + \frac{1}{c} \frac{\partial E^{(0)}}{\partial t} \]

\[ \nabla \cdot B^{(1)} = 0 \]

**Induced Electric fields / Back Emf**

\[ \nabla \cdot D^{(2)} = 0 \]

\[ -\nabla \times E^{(2)} = \frac{1}{c} \frac{\partial B^{(1)}}{\partial t} \quad \text{could call} \quad E^{(2)} = E^{\text{ind}} \]

Want to compute the energy stored in magnetic field

Back Emf: Imagine slowly increasing the current. Changing the current makes a changing magnetic field inducing a Back Emf. The work the battery does to increase the current is the energy stored in the fields.
Induction pg. 2

almost

Take \( \gamma \) magneto statics, ie \( \mathbf{D}^{(o)} = 0 \).

\[
\nabla \times \mathbf{H} = -\mathbf{j},
\]

\[
\nabla \cdot \mathbf{B} = 0
\]

And

\[
-\nabla \times \mathbf{E}^{\text{ind}} = \frac{1}{c} \mathbf{j} \times \mathbf{B}
\]

Then the work by battery is

\[
\frac{\delta U}{\delta t} = \frac{\delta W_{\text{batt}}}{\delta t} - \oint_{S_t} \mathbf{j} \cdot \delta \mathbf{E}^{\text{ind}} / \delta t
\]

\[
\quad = -\oint \nabla \times \mathbf{H} \cdot \delta \mathbf{E}^{\text{ind}}
\]

\[
\quad = -\oint \nabla \cdot (\mathbf{H} \times \mathbf{E}) \cdot \delta \mathbf{E}^{\text{ind}} - \mathbf{H} \cdot \nabla \times \delta \mathbf{E}
\]

\[
\quad = -\oint \frac{\partial}{\partial t} \mathbf{H} \cdot \delta \mathbf{E}^{\text{ind}}
\]

\[
\frac{\delta U}{\delta t} = \oint \mathbf{H} \cdot \delta \mathbf{B}
\]

\[
\delta U = \oint \mathbf{H} \cdot \delta \mathbf{B}
\]
Induction pg. 3

Then for linear media \( SB = \mu SH \)

\[
U = \frac{1}{2} \int \frac{H^2}{\mu} = \left[ \int \frac{H \cdot \nabla \times A}{c} \, d^3x \right]_V = U
\]

These equations are often expressed in terms of \( J \) and \( \overrightarrow{A} \) rather than \( \overrightarrow{B} \)

Indeed,

\[
\delta U = \int_V \nabla \cdot \delta \overrightarrow{B} \quad \delta E^{\text{ind}} = -1 \omega \varepsilon \overrightarrow{S} \overrightarrow{A} \cdot \overrightarrow{S} \left( \overrightarrow{A} \right) \cfrac{1}{c}
\]

\[
\delta U = \int_V \nabla \times H \cdot \delta \overrightarrow{A} \quad \text{By parts (no minus)}
\]

\[
\delta U = \int_V \nabla \times H \cdot \delta \overrightarrow{A} \quad \text{by cross product}
\]

\[
\delta U = \int_V \nabla \cdot \delta \overrightarrow{A} \quad \text{by cross product}
\]

For linear media \( \overrightarrow{S} \overrightarrow{A} \propto \mu \overrightarrow{S} \overrightarrow{J} \)

\[
U = \frac{1}{2} \int \frac{\overrightarrow{J} \cdot \overrightarrow{A}}{c} \, d^3x
\]
Wires pg. 1

**Inductance in Wires**

1. \[ U_B = \frac{1}{\mu} \int \frac{j \cdot A}{\sqrt{V}} d^3x = \frac{1}{\mu} \int \frac{\vec{H} \cdot \vec{B}}{\sqrt{V}} \]

   \( U_B \) is a property of state

2. \[ S U_B = \int \frac{j \cdot \delta A}{\sqrt{c}} \]

For a set of wires: \[ \int j \cdot d^3x = I d \]

\[ \bigcirc \quad \begin{array}{c} I_1 \end{array} \quad \begin{array}{c} \text{I}_2 \end{array} \quad \begin{array}{c} I_3 \end{array} \]

Then find summed over \( a \) = loops

1. \[ U = \frac{1}{\mu} \int \frac{I}{c} \cdot \Phi_a \]

   \[ \Phi_a = \int A \cdot dl = \int B \cdot dA \]

   \( a \)-th loop

2. \[ S U = \frac{1}{\mu} \int \frac{S \Phi_a}{c} \]

   flux through \( a \)-th loop

Note that, \[ \hat{A}(x) = \rho \int \frac{\delta(x-x_0)}{\sqrt{4\pi |x-x_0|}} \]

\[ U_B = \frac{\mu}{2} \int \frac{d^3x \cdot d^3x}{\sqrt{V}} \frac{j(x)}{c} \cdot \frac{\delta(x)}{c} \frac{j(x)}{c} \]

\[ \frac{4\pi}{\sqrt{|x-x_0|}} \]
So for a set of wires

\[ U = \frac{1}{2} \mathbf{I}_a \mathbf{M}_{ab} \mathbf{I}_b \]

inductance matrix

\[ \mathbf{M}_{11} \] is the self inductance of the first loop

\[ \mathbf{M}_{12} \] is the mutual inductance between the 1st & 2nd

Then since \[ U_B = \frac{1}{2} \mathbf{I}_a \mathbf{\phi}_a \]

\[ \frac{\mathbf{\phi}_a}{c} = \mathbf{M}_{ab} \mathbf{I}_b \] \( \text{back emf} \)

And for any circuit \[ E_a = -L d\mathbf{\phi}_B = - \mathbf{M}_{ab} d\mathbf{I}_b \]

\[ \frac{dt}{c} \]
Problem on Mutual Inductance and Force

Compute the mutual inductance of a ring and a long straight wire

\[ R \]

\[ \bigcirc \quad I_1 \]

\[ D \]

\[ \downarrow \quad I_2 \]

Solution

Current in wire one

\[ U_{12} = \int \frac{I_1}{c} \cdot \mathbf{A}_2 \]

\[ V \]

\[ = \frac{I_1}{c} \int \mathbf{A}_2 \cdot d\mathbf{l}_1 = \frac{I_1}{c} \int \mathbf{B}_2 \cdot d\mathbf{A}_1 \]

Then the field from the wire is

\[ \mathbf{B}_2 = \frac{I_2}{2\pi} \frac{1}{\rho} \hat{\phi} \]

So we need to integrate this field from the wire over the area of the ring.
Mutual Inductance and Force pg. 2

points out given by circulation of $I_1$ (see picture)

$$\int_{R^2} \mathbf{B} \cdot d\mathbf{a} = \frac{I_1}{c} \left( \int_{2\pi}^{2\pi} \frac{2(R^2 - (p-D)^2)^{1/2}}{2\pi p} dp \right)$$

points out

We have

$$U_{12} = \int_{D-R}^{D+R} dp \frac{I_1 I_2}{c^2} \left[ \frac{2(R^2 - (p-D)^2)^{1/2}}{2\pi p} \right]$$

$$U_{12} = \frac{I_1 I_2}{c^2} \left[ \frac{D - \sqrt{D^2 - R^2}}{c^2} \right]$$

So

$$M_{12} = \frac{1}{c^2} \left( D - \sqrt{D^2 - R^2} \right)$$

Then we might want to compute the force between the ring and the wire. To do this we ask about the change in $U_B$, as the distance between the ring and the wire is changed:

$$SU_3 = \int_a^{a+\Delta a} + \Delta W_{\text{mech}}$$

$$\Delta W_{\text{mech}} = -\int_{\text{ring}} F \cdot \vec{SD}$$

change in energy $\Delta W_{\text{batt}}$

stored in fields $\Delta W_{\text{batt}}$ work done by $\bar{F} = -\bar{\mathcal{E}}$ force on ring

work done by battery to keep current fixed mechanically applied
Mutual Inductance and Force pg. 3

\[ S \mathbf{u} = \frac{1}{2} I_a \delta m_{ab} I_b \]

\[ I_a \delta \Phi_a = I_a \delta m_{ab} I_b \]

So

\[ S \mathbf{u}_b - I_a \delta \Phi_a = -\frac{1}{2} I_a \delta m_{ab} I_b = -F \delta \mathbf{D} \]

So

\[ F_x = + \frac{\delta m_{ab} I_a I_b}{S D \frac{c^2}{2}} = \frac{I_1 I_2}{c^2} \left( 1 - \frac{D}{\sqrt{D^2 - R^2}} \right) \]

\[ F_y = -\frac{I_1 I_2}{c^2} \left( \frac{D}{\sqrt{D^2 - R^2}} - 1 \right) \]

indicates an attractive force

i.e. force in negative x-direction

\[ \frac{F}{I_1 I_2 / c^2} \rightarrow \text{attractive force} \]