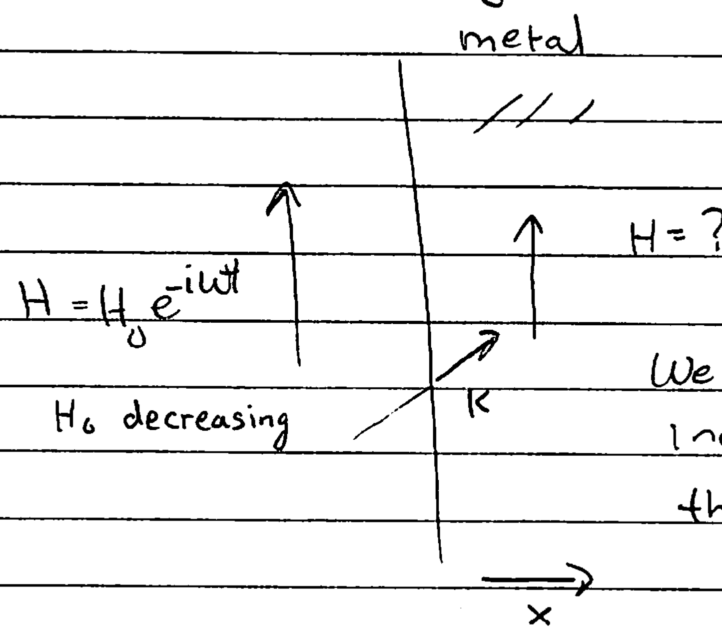
Quasi-Statics + Induction in Metals

- diffusion of magnetic fields



We will find that induced currents cause the magnetic fields to decay in metal

① If the magnetic fields are increasing (as drawn) which way do the currents flow?

② What are the dimensionful parameters?

$$H_0, \quad \omega, \quad c, \quad \sigma$$

Then

$$[\sigma] \sim \frac{1}{s} \qquad \sigma \sim 10^{18} \text{ Hz for Cu}$$

We will see that a characteristic scale for decay is δ

$$\delta = \sqrt{\frac{2c^2}{\sigma\omega}} = \left(\frac{(m/s)^2}{1/s \;\; 1/s}\right)^{1/2} \sim m$$

Analysis of Quasi-statics in metals

$\nabla \cdot E = 0$

$\nabla \times H = j^{ind}$

$\nabla \cdot B = 0$

$-\nabla \times E = \frac{1}{c} \partial_t \vec{B}$

So $j^{ind} = \sigma E^{ind}$ then we have with $B = \mu H$

$\nabla \times H = \frac{\sigma}{c} E^{ind}$

$\nabla \times \nabla \times H = \frac{\sigma}{c} \nabla \times E^{ind}$ $\quad \nabla \times E = \frac{\mu}{c} \partial_t H$

$\nabla (\cancel{\nabla \cdot H}) - \nabla^2 H = -\sigma \frac{\mu}{c^2} \partial_t H$

So find a diffusion equation for magnetic fields:

$$\boxed{\nabla^2 H = \frac{\sigma \mu}{c^2} \partial_t H}$$

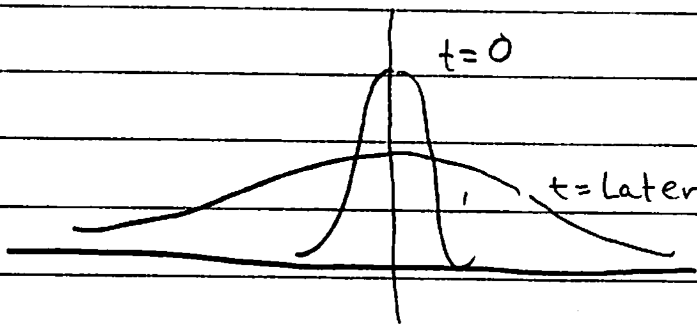
Diffusion equation

$$\boxed{\partial_t n = D \nabla^2 n}$$

canonical form

diffusion coefficient

A primer on Diffusion Equation:



A drop of dye in water
The width of the drop increases in time

$$(\Delta X)^2 = 2Dt$$

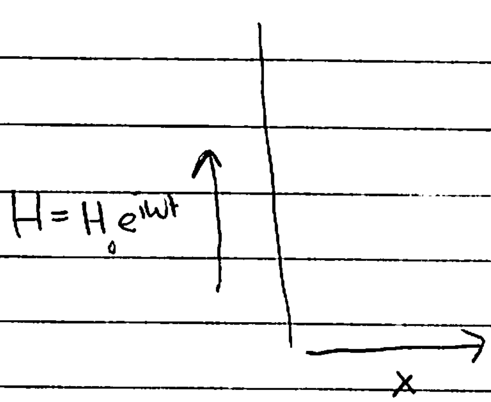
diffusion coefficient

- The diffusion equation smears out features
- The magnetic diffusion coefficient:

$$D = \frac{c^2}{\mu\sigma} \qquad \mu \text{ is dimensionless}$$

$$\simeq \frac{1\,\text{cm}^2}{\text{millisec}} \qquad \text{for Cu } \mu = 1 \;\; \sigma \simeq 10^{18}\,\text{Hz}$$

Solving the diffusion equation



try

$$\vec{H}(x,t) = H_0\, e^{-i\omega t}\, h(x)\, \hat{z}$$

Then substitute into

$$\nabla^2 H = \frac{1}{D}\partial_t H$$

Solving the Diff Eq. pg. 2

Then find $\partial_t H \propto -i\omega H$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{i\omega}{D}\right) h(x) = 0$$

So try $h(x) = c\, e^{ikx}$:

$$-k^2 + \frac{i\omega}{D} = 0 \implies k_\pm = \pm (1+i)\sqrt{\frac{\omega}{2D}} = \pm \frac{(1+i)}{\delta}$$

Note $\pm\sqrt{i} = \pm\frac{(1+i)}{\sqrt{2}}$

δ is skin depth
see below

$$\delta = \sqrt{\frac{2D}{\omega}}$$

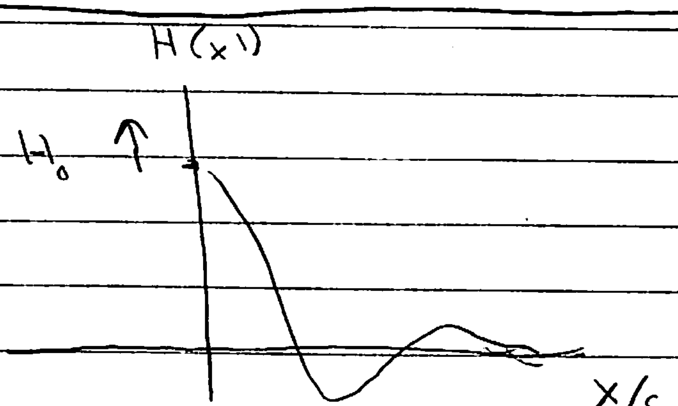
Thus, $e^{ik_+ x} = e^{ix/\delta}\, e^{-x/\delta}$

while $e^{-ik_- x} = e^{-ix/\delta}\, e^{+x/\delta}$ ← Discard

So

$$H(x,t) = \mathrm{Re}\left(H_0 e^{-i\omega t} e^{ix/\delta - x/\delta}\right)$$

$$\boxed{H(x,t) = H_0\, e^{-x/\delta} \cos(x/\delta - \omega t)}$$



Thus find that the magnetic, field decays with characteristic length, δ.

$$\delta = \sqrt{\frac{2D}{\omega}} = \sqrt{\frac{2c^2}{\omega\mu\sigma}} \qquad \sigma \sim 10^{18}\ Hz$$

For $D_{Cu} \sim \frac{cm^2}{millisec}$ find $\delta \sim cm \frac{1}{\sqrt{f_{kHz}}}$

property of metal

property of metal and probe

We can calculate the electric field

$$\frac{j^x}{c} = \frac{\sigma E}{c} = \nabla \times B$$

Find for B in z-direction

$$\frac{j^y}{c} = -\frac{\partial B^z}{\partial x} = Re\left[-\frac{\partial}{\partial x} H_0 e^{-i\omega t} e^{ik_+ x}\right]$$

$$= Re\left[-ik_+ H_0 e^{-i\omega t} e^{ik_+ x}\right]$$

$$\frac{j^y}{c} = \frac{\sqrt{2}}{\delta} H_0 e^{-x/\delta} \cos(x/\delta - \omega t - \pi/4)$$

Analysis of Diffusion Eq. pg. 1

① So a parametric estimate for E^{ind} is:

$$E^{ind} \sim \frac{j/c}{\sigma/c} \sim \frac{cH_0}{\delta\sigma} \qquad \delta \equiv \sqrt{\frac{2c^2}{\omega\mu\sigma}}$$

$$\mu \sim 1$$

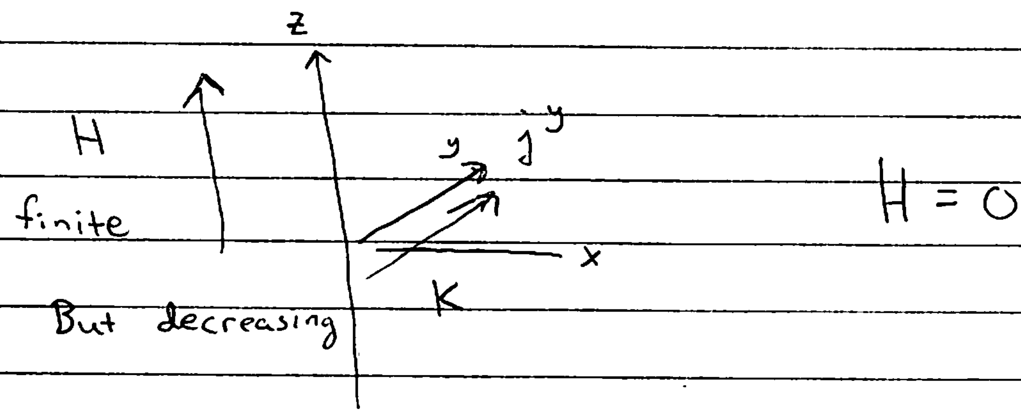
$$E^{ind} \sim \sqrt{\frac{\omega}{\sigma}}\, H_0 \qquad \nabla \times B = j^{ind} + \cancelto{0}{\frac{1}{c}\partial_t E^{ind}}$$

So for E^{ind} to be small (which we assumed), we must have

$$\omega \ll \sigma \qquad \sigma_{Cu} \sim 10^{18}\ Hz$$

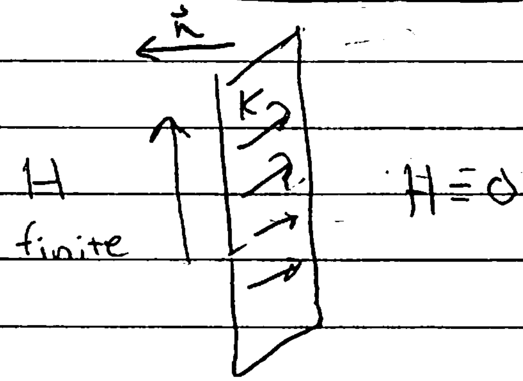
which is satisfied deap into optical frequencies.

② Lets compute the total current



If I look at this from far away, what do I see

Analysis of Diffusion Eq. pg 2



← I see this
(I cant see the boundary layer of width ~ δ)

$$\frac{K^y}{c} = \int_0^\infty dx \, \frac{\sqrt{2}}{\delta} H_0 e^{-x/\delta} \cos\left(\frac{x}{\delta} - \omega t - \pi/4\right)$$

do integral

$$\frac{K^y}{c} = H_0 \cos \omega t$$

This is what you expect from boundary conditions

$$\vec{n} \times (\vec{H}_{out} - \cancel{\vec{H}_{in}}^{\,0}) = \frac{\vec{K}}{c}$$

$$+ n \times \vec{H}_{out} = \frac{\vec{K}}{c} \qquad H_{out} = H_0 \cos \omega t \ \hat{z}$$

$$H_0 \cos \omega t = \frac{K^y}{c}$$