\[ \nabla \times B = j_{\text{mat}} + j_{\text{ext}} / c \]

\[ \nabla \cdot B = 0 \]

Then we wrote down a constituent relation:

\[ j_{\text{mat}} = \sigma \frac{\partial B}{\partial t} + \mu_0 \nabla \times B + \chi_m B \nabla \times B + \text{higher} \]

\[ \text{parity} \quad \text{parity} \]

\[ \text{odd} \quad \text{odd} \]

\[ j_{\text{mat}} = \chi_m^B \nabla \times B \implies j_{\text{mat}} = \frac{\nabla \times m}{c} \]

Then

\[ M = \chi_m^B \vec{B} \]

\[ \nabla \times B = \nabla \times M + j_{\text{ext}} / c \]

\[ \nabla \cdot B = 0 \]

Or

\[ \nabla \times (B - M) = j_{\text{ext}} / c \]

\[ \nabla \cdot B = 0 \]
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Usually expressed as \( M(H) \) rather than \( B \).

Using,

\[
H = B - M \\
H = B - x^B \cdot B
\]

\[
1 \quad H = B \quad (1 - x^B)
\]

i.e.

\[
\boxed{M \cdot H = B}
\]

where \[ \mu = \frac{1}{(1 - x^B)} \]

We also recall the defining relation

\[
\nabla \times M = \frac{\dot{j}_{\text{mat}}}{\epsilon}
\]
Linear Magnetic Materials

- Diamagnetic (oppose $\equiv$ dia)  $\vec{M} = \chi_m^{B}\vec{B}$

  $\chi_m^B < 0$ and $\mu < 1$ find $\chi_m^B \approx 10^{-5}$

  Typically, this is related to orbital motion of electrons, with all spins paired. The orbits change to oppose the change in flux.

- Paramagnetic (same $\equiv$ para)

  Typically related to spin aligning with the magnetic field.

  $\uparrow$  $\uparrow$  $\uparrow$ 

  $\chi_m^B > 0$ and $\mu > 1$  $\chi_m^B \approx 10^{-5}$

Let us understand the order of magnitude of $\chi_m^B$ for diamagnetic and paramagnetic substances.
Dimensional Analysis of Linear Magnetic Substances

\[ \mathbf{J} = \kappa \hat{m} \mathbf{c} \mathbf{\nabla} \times \mathbf{B} \]

Then, dimensions give

\[ [\mathbf{\nabla} \times \mathbf{B}] = \frac{\mathbf{q}}{m^2 \mathbf{m}} \]

\[ [\mathbf{F}] = \frac{\mathbf{q}}{m^2 \mathbf{s}} \quad \text{naive dimension} \]

So,

\[ [c \kappa \hat{m}] = \frac{\mathbf{m}}{\mathbf{s}} \quad \text{so expect that} \quad c \kappa \hat{m} \sim \mathbf{v}_{\text{micro}} \]

Where \( \mathbf{v}_{\text{micro}} \) is the typical electron velocity.

In fact, we can anticipate that since the forces which generate the currents \( \mathbf{F} = \mathbf{q} (\mathbf{v}/c) \times \mathbf{B} \) are small (since \( \mathbf{v}_{\text{micro}}/c \ll 1 \)), the currents \( \kappa \) are smaller than the naive dimension by \( \mathbf{v}/c \), i.e.

\[ c \kappa \hat{m} \sim \mathbf{v}_{\text{micro}} \left( \frac{\mathbf{v}_{\text{micro}}}{c} \right) \]

And thus

\[ \kappa \hat{m} \sim \left( \frac{\mathbf{v}_{\text{micro}}}{c} \right)^2 \]

Compare to the electric case \( \kappa_e \sim 1 \)
To estimate \( V \) we recall the Bohr model.

The Bohr model can be remembered by the slogan "\( \beta = \alpha \)"

\[
\beta = \frac{V_{\text{bohr}}}{c} \Rightarrow \alpha = \frac{e^2}{4\pi \hbar c} = 1
\]

Also useful:

\[
13.6\text{eV} = \frac{1}{2} PE = \frac{1}{2} \left( \frac{e^2}{4\pi \alpha_0} \right) = KE = \frac{\hbar^2}{2m_e^2} = \frac{(mc^2)^2}{2}
\]

Thus expect that

\[
\chi_m^2 \sim \alpha^2 \sim 10^{-5}
\]

This works fine for linear substances. For ferromagnetic materials, the magnetization can be much larger, and is usually non-linear.

\[
\vec{B} = \mu(\vec{H}) \vec{H}
\]

\( C \) can be like \( 10^3 \)

Why? Because ferromagnetic substances involve all the atoms working cooperatively, even in the absence of external fields.
The spins tend to align in ferromagnetic substances because if the spin wave-function is symmetric, then the spatial wave-function can be anti-symmetric, minimizing the Coulomb energy. This is a much larger effect (by $(\nu/c)^2$) than the dipole-dipole interaction, which would cause the spins to anti-align. In real ferromagnets, the domains grow until the magnetic interaction competes with the short-range Coulomb interaction.
Non-linear magnetic material & Hard Ferromagnets

In ferromagnets the induced magnetization depends non-linearly on $\mathbf{B}$. Assume $\mathbf{B}$ only vector parity odd (throw away)

$$\mathbf{j}^{\text{mat}}(\mathbf{k}) = \frac{1}{2} \mathbf{B}(k) \left( \sigma_{B}^{k} + \frac{1}{2} \mathbf{B}^{2}(k) + \frac{1}{2} (\mathbf{B}^{2}(k))^{2} + \ldots \right)$$

$$+ i k \times \mathbf{B}(k) \left( x_{m}^{B} + c_{1} \mathbf{B}^{2}(k) + c_{2} (\mathbf{B}^{2}(k))^{2} + \ldots \right)$$

+ higher derive

Thus, reasonably generally, one finds a constitutive relation

$$\mathbf{j}^{\text{mat}} = \nabla \times \mathbf{M}(\mathbf{B})$$

$a$ non-linear function of

after Fourier transforming back to coordinate space.

Thus the macroscopic equations for magneto-statics read

$$\nabla \times \mathbf{B} = \nabla \times \mathbf{M}(\mathbf{B}) + j_{\text{ext}} / \varepsilon$$

$$\nabla \cdot \mathbf{B} = 0$$

$\mathbf{M}(\mathbf{B})$ needs to be specified and generally gives rise to very non-linear equations.
One case that can be handled is that of hard ferromagnets where \( M(x) \) is a fixed function of space.

\[
\mathbf{j}(x) = \nabla \times M(x)
\]

The boundary conditions still apply, namely

\[
\mathbf{n} \times (\mathbf{H}_{out} - \mathbf{H}_{in}) = \mathbf{K}_{\text{free}} / c
\]

\[
\mathbf{n} \cdot (\mathbf{B}_{out} - \mathbf{B}_{in}) = 0
\]

\[
\mathbf{n} \times (\mathbf{m}_2 - \mathbf{m}_1) = \mathbf{K}_{\text{mut}} / c
\]
Example

- A uniformly magnetized rod of height $h$, radius $a$, and magnetization $\mathbf{M} = M_0 \hat{z}$. Determine the magnetic field on axis.

\[
\mathbf{J} = \nabla \times \mathbf{M} = 0 \quad \text{inside and out}
\]

But we have boundary conditions which give a surface current

Using

\[
\mathbf{n} \times (\mathbf{M}^\text{out} - \mathbf{M}^\text{in}) = \mathbf{k} \quad \text{at } \mathbf{n}
\]

\[
- \mathbf{n} \times (M_0 \hat{z}) = \mathbf{k} \quad \text{at } \mathbf{n}
\]

\[
M_0 \varphi = k \quad \text{at } \mathbf{n}
\]

So we find a cylinder of current. From a ring of width $dx$

\[
\frac{d\mathbf{B}_{\text{z}}}{2} = \frac{d\mathbf{J}}{2} \frac{a^2}{(a^2 + (z+x)^2)^{3/2}}
\]

\[
\frac{d\mathbf{J}}{2} = \frac{k}{c} dx = M_0 dx
\]
\[ B_z = \int_0^h dx \frac{M}{2} \frac{a^2}{(z+x)^2 + a^2 + z^2} \]

\[ B_z = \frac{M}{2} \left[ \frac{(h+z)}{(a^2 + (h+z)^2)^{\frac{3}{2}}} - \frac{z}{(a^2 + z^2)^{\frac{3}{2}}} \right] \]

\[ B_z = \frac{M}{2} \left[ \cos \theta_1 - \cos \theta_2 \right] \]