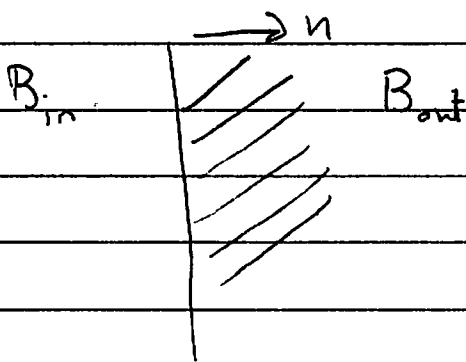


## Last Times

$$\nabla \times \mathbf{B} = \mathbf{j}/c$$

$$\nabla \cdot \mathbf{B} = 0$$

Write Down B.C



$$\vec{n} \times (\vec{B}_{out} - \vec{B}_{in}) = \vec{K}/c$$

$$\vec{n} \cdot (\vec{B}_{out} - \vec{B}_{in}) = 0$$

This can also be written for  $\vec{A}$

$$-\nabla^2 \vec{A} = \vec{j}/c$$

$$\vec{B} = \nabla \times \vec{A}$$

Question:

• What is the vector potential corresponding to a const  $\vec{B}$  field?

Ans:

$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$$

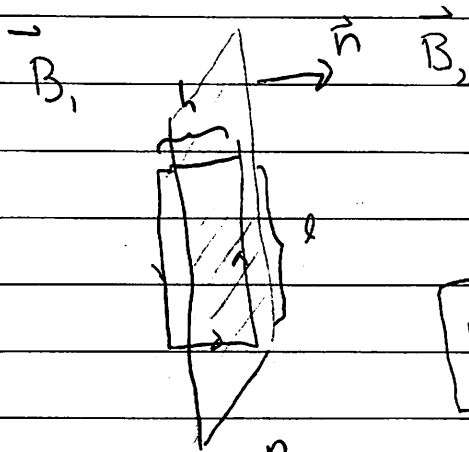
Prf

$$(\nabla \times \vec{A})_i = \epsilon_{ijk} \partial_j \epsilon_{klm} \frac{B^l r^m}{2}$$

$$= \epsilon_{ijk} \epsilon_{lmk} \delta_j^m \frac{B^l}{2} = \overbrace{\epsilon_{ijk} \epsilon_{lyk}}^{2\delta_{il}} \frac{B^l}{2} = B_i$$

## Boundary Conditions

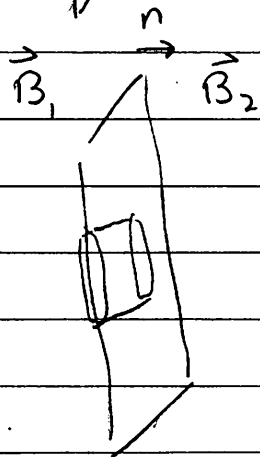
Before we can actually solve the eqns we need B.C



current per area

$$\int \vec{B} \cdot d\vec{\ell} = \frac{K}{c} \cdot hl$$

$$\vec{n} \times (\vec{B}_2 - \vec{B}_1) = \frac{K}{c} \Rightarrow B_2^\perp - B_1^\perp = K$$



$$\int (\nabla \cdot \vec{B}) = 0$$

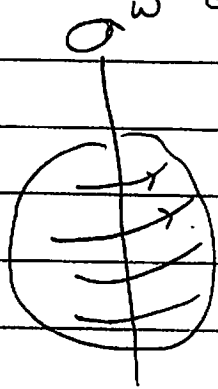
pill box

$$\Rightarrow \vec{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0$$

$$B_2^\perp - B_1^\perp = 0$$

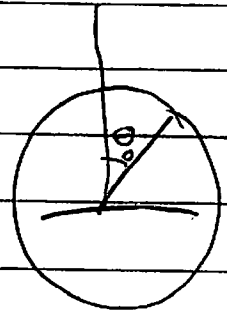
- The parallel components jump, while the perpendicular components change smoothly

Example - Rotating Charged Sphere of Radius  $a$ , and charged  $Q$ .



$$\vec{v} = \vec{\omega} \times \vec{x}$$

$$\vec{v} = \omega a \sin \theta \hat{\phi}$$



Now

$$\vec{j} = \rho \vec{v}$$

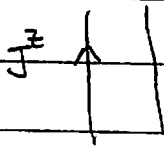
$$\rho = \sigma \delta(r_0 - a)$$

$$= \sigma \delta(r_0 - a) \omega a \sin \theta \hat{\phi}$$

$$\uparrow \frac{Q}{4\pi a^2}$$

### Methods of Solution

- Direct use of  $-\nabla^2 \vec{A} = \vec{j}$ . In general complicated except for simple geometries, e.g. for 2D problems with azimuthal symmetry and no  $z$  dependence, where only  $A^z$  is non-zero



$$-\nabla^2 A^z = J^z(\rho, \phi)$$

$$\left[ -\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right] A^z = J^z$$

- Magnetic scalar potential (next hour)

## Rotating Sphere 2

### Direct Integration

$$\vec{A}(\vec{r}) = \int d^3r_0 \frac{\vec{j}(\vec{r}_0)}{4\pi |\vec{r} - \vec{r}_0|}$$

A good choice here:

$$(\star) A(\vec{r}) = \int r_0^2 dr_0 d\Omega_0 \frac{\sigma \delta(r_0 - a) \sin\theta_0 \cos\phi_0 \hat{\phi}_0}{4\pi |\vec{r} - \vec{r}_0|}$$

Now look outside sphere:  $r > r_0$

$$\star\star \frac{1}{4\pi |\vec{r} - \vec{r}_0|} = \sum_{lm} \frac{1}{(2l+1)} \frac{r_0^l}{r^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta_0, \phi_0)$$

where  $\vec{r}_0 = (r_0 \sin\theta_0 \cos\phi_0, r_0 \sin\theta_0 \sin\phi_0, r_0 \cos\theta_0)$ . Then look at

$$\sin\theta_0 \hat{\phi}_0 = \underbrace{\sin\theta_0 \cos\phi_0}_{\propto Y_{1,0} \text{ and } Y_{1,-1}} \hat{i} + \underbrace{\sin\theta_0 \sin\phi_0}_{\propto Y_{1,1} \text{ and } Y_{1,-1}} \hat{j}$$

Substituting Eq. ( $\star\star$ ) into  $\star$  we find integrals like:

$$\int d\Omega_0 Y_{lm}^*(\theta_0, \phi_0) (\sin\theta_0 \cos\phi_0) = 0 \text{ unless } l=1$$

Thus only  $l=1$  survives integrations  $\int d\Omega_0$

## Rotating Sphere 3

Comparison with the cartesian form

$$\begin{aligned} \frac{1}{4\pi |\vec{r} - \vec{r}_0|} &= \frac{1}{(r^2 + r_0^2 - 2\vec{r} \cdot \vec{r}_0)^{1/2}} \\ &= \underbrace{\frac{1}{4\pi r}}_{l=0} + \underbrace{\frac{\vec{r} \cdot \vec{r}_0}{4\pi r^2}}_{l=1} + \underbrace{O(r^{-2})}_{l=2} \end{aligned}$$

We see that inside the integral can replace

$$\frac{1}{4\pi |\vec{r} - \vec{r}_0|} \longrightarrow \frac{1}{4\pi} \frac{\vec{r} \cdot \vec{r}_0}{r^3}$$

So

$$\vec{A} = \frac{a^2 \sigma}{c} \int d\Omega_0 \frac{1}{4\pi} \frac{\vec{r} \cdot \vec{r}_0}{r^3} \underbrace{(\vec{\omega} \times \vec{r}_0)}_{\omega \sin \theta_0 \hat{\phi}}$$

So look at  $i$ -th component of this

$$\vec{r} \cdot \vec{r}_0 (\vec{\omega} \times \vec{r}_0)^i = r(r_0)_\ell \varepsilon^{ijk} \omega_j (r_0)_k$$

## Rotating Sphere 4

So

$$A^i = \frac{a^2 \sigma}{c} \frac{\epsilon^{ijk} \omega_j r^l}{4\pi r^3} \int d\Omega_0 (r_0)_k (r_0)_l$$

$$I = \frac{4\pi a^3}{3} \delta_{kl}$$

This integral follows as so:

$$\int d\Omega_0 (r_0)_k (r_0)_l = 0 \quad \text{for } k \neq l \quad \int_{\text{sphere}} z y = 0$$

$$\text{But } \int (r_0)_1^2 d\Omega_0 = \int (r_0)_2^2 d\Omega_0 = \int (r_0)_3^2 d\Omega_0$$

$\sim \langle x^2 \rangle \quad \sim \langle y^2 \rangle \quad \sim \langle z^2 \rangle$

So

$$\frac{1}{3} \int d\Omega_0 \underbrace{\vec{r}_0 \cdot \vec{r}_0}_{= a^2} = \frac{4\pi a^3}{3}$$

$$(r_0)_1^2 + (r_0)_2^2 + (r_0)_3^2 = a^2$$

## Rotating Sphere 5

Thus we find:

$$A^i = \frac{a^2 \sigma}{c} \frac{\epsilon^{ijk}}{4\pi r^3} \omega_j r^k \left( \frac{4\pi a^2}{3} \delta_{ka} \right)$$

using  $-\sigma = Q/4\pi a^2$

$$A^i = \frac{1}{4\pi} \left[ \left( \frac{Qa^2 \vec{\omega}}{3c} \right) \times \vec{r} \right]^i / r^3$$

$$= \vec{m}$$

← magnetic dipole moment

So:

$$\vec{A} = \frac{1}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} \quad r > a \quad (\text{outside})$$

or  $\vec{B} = (3(\vec{n} \cdot \vec{m})\vec{n} - \vec{m}) / 4\pi r^3$

Thus we see that the magnetic field outside the sphere is that of a magnetic dipole. An entirely similar calculation for  $\vec{A}$  inside the sphere shows that:

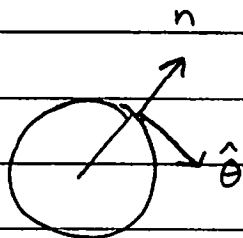
$$\vec{A}_{in} = \frac{1}{4\pi} \left( \frac{\vec{m}}{a^3} \right) \times \vec{r} \quad r < a \quad (\text{inside})$$

This is the vector potential of a constant  $\vec{B}$  field,  
 $\vec{B}_{in} = \frac{2\vec{m}}{4\pi(a^3)}$

## Rotating Sphere - Check BC

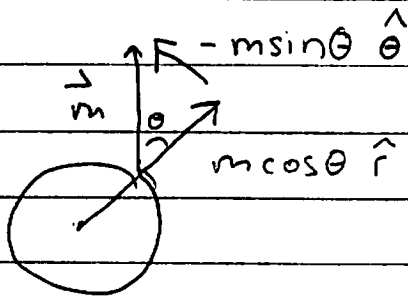
We will now check that the B.C. are satisfied.

$$\vec{n} \times (\vec{B}_{out} - \vec{B}_{in}) = \frac{\vec{K}}{c}$$



$$(\vec{B}_{out})_{\theta} - (\vec{B}_{in})_{\theta} = \frac{K}{c} \phi$$

Using:



$$-m \sin \theta \hat{\theta}$$

$$m \cos \theta \hat{r}$$

this comes from  $-\vec{m}$  of

$$\vec{B}_{out}^{\theta} = \frac{1}{4\pi a^3} (+m \sin \theta) \quad \leftarrow \quad \frac{3(n \cdot \vec{m})n - \vec{m}}{4\pi r^3}$$

$$\vec{B}_{in}^{\theta} = \frac{2(-m \sin \theta)}{4\pi a^3}$$

So

$$\vec{B}_{out}^{\theta} - \vec{B}_{in}^{\theta} = \frac{3m \sin \theta}{4\pi a^3}$$

$$m = \frac{Qa^2 \omega}{3c} \quad \sigma = \frac{Q}{4\pi a^2}$$

$$= \sigma \omega a \sin \theta$$

↑ surface current ✓