13.1 Basic equations

(a) We wrote down the wave equations in the covariant gauge:
\[
-\Box \Phi = \rho (t_o, r_o) \tag{13.1}
\]
\[
-\Box A = J (t_o, r_o)/c \tag{13.2}
\]

(b) Then we used the green function of the wave equation
\[
G(t, r | t_o, r_o) = \frac{1}{4\pi |r - r_o|} \delta(t - t_o + \frac{|r - r_o|}{c}) \tag{13.3}
\]
to determine the potentials (\(\Phi, A\)) with the current
\[
\frac{J^\mu}{c} = (\rho, \frac{J}{c}) = (q \delta^3 (r_o - r_*(t_o)), q \frac{v(t_o)}{c} \delta^3 (r_o - r_*(t_o))) \tag{13.4}
\]
This yields the Lienard-Wiechert potentials
\[
\Phi = \frac{q}{4\pi|r - r_*(T)|} \frac{1}{1 - n \cdot \beta(T)} \Rightarrow \frac{q}{4\pi r} \frac{1}{1 - n \cdot \beta(T)} \tag{13.5}
\]
\[
A = \frac{q}{4\pi|r - r_*(T)|} \frac{\beta(T)}{1 - n \cdot \beta(T)} \Rightarrow \frac{q}{4\pi r} \frac{\beta(T)}{1 - n \cdot \beta(T)} \tag{13.6}
\]
where the retarded time is
\[
T(t, r) = t - \frac{|r - r_*(T)|}{c} \Rightarrow T(t, r) = t - \frac{r}{c} + \frac{n \cdot r_*(T)}{c} \tag{13.7}
\]
The terms after the Longrightarrow indicate the far field limit

(c) The Lienard Wiechert potential can also be obtained by integrating over \(r_o\) in Eq. (11.8).

(d) The factor “collinear factor” (my name), or \(dT/dt\)
\[
\frac{dT}{dt} = \frac{1}{(1 - n \cdot \beta)} \tag{13.8}
\]
\[
\frac{dT}{dr^i} = \frac{1}{(1 - n \cdot \beta)} \frac{-n_i}{c} \tag{13.9}
\]
is quite important. We gave a physical interpretation of it in class. If a wave form is observed to have a time scale of \(\Delta t\), then the formation time of the wave, \(\Delta T\), is
\[
\Delta T = \frac{dT}{dt} \Delta t = \frac{\Delta t}{1 - n \cdot \beta} \tag{13.10}
\]
In particular, a fourier component with frequency \(\omega\) in the observed wave was formed over the time
\[
\Delta T \sim \frac{1}{\omega (1 - n \cdot \beta)} \tag{13.11}
\]
(e) In the ultrarelativistic limit \(1 / (1 - \beta \cos \theta)\) is often approximated for \(\theta \ll 1\) and for ultra-relativistic particles \(1 - \beta \simeq 1 / 2 \gamma^2\)

\[
\frac{1}{1 - \mathbf{n} \cdot \mathbf{\beta}} = \frac{2 \gamma^2}{1 + (\gamma \theta)^2}
\]

(13.12)

(f) The magnetic and electric fields can be determined from \(\mathbf{E} = -\frac{1}{c} \partial_t \mathbf{A}_{\text{rad}} - \nabla \Phi\). As discussed in a separate note ("retarded_time.pdf"), in the far field limit this is the same as computing

\[
\mathbf{E}(t, r) = \mathbf{n} \times \mathbf{n} \times \frac{1}{c} \partial_t \mathbf{A}_{\text{rad}}(T)
\]

(13.13a)

\[
= \mathbf{n} \times \mathbf{n} \times \frac{1}{1 - \mathbf{n} \cdot \mathbf{\beta}} \frac{1}{c} \frac{\partial}{\partial T} \mathbf{A}_{\text{rad}}(T)
\]

(13.13b)

\[
= \frac{1}{1 - \mathbf{n} \cdot \mathbf{\beta}} \frac{1}{c} \frac{\partial}{\partial T} \left[ \frac{q}{4\pi r} \frac{\mathbf{n} \times \mathbf{n} \times \mathbf{\beta}}{1 - \mathbf{n} \cdot \mathbf{\beta}} \right]_{\text{ret}}
\]

(13.13c)

\[
= \frac{q}{4\pi c^2} \left[ \mathbf{n} \times (\mathbf{n} - \mathbf{\beta}) \times \mathbf{a} \right] \frac{1}{(1 - \mathbf{n} \cdot \mathbf{\beta})^3} \left( \frac{\partial}{\partial T} \right)_{\text{ret}}
\]

(13.13d)

The \([\cdot]_{\text{ret}}\) indicates that the velocity and acceleration are to be evaluated at the retarded time \(T(t, r)\).

The magnetic field is

\[
\mathbf{B} = \mathbf{n} \times \mathbf{E}
\]

(13.14)

For below, it is worth noting below that

\[
\frac{1}{c} \frac{\partial}{\partial T} \left[ \mathbf{n} \times \mathbf{n} \times \mathbf{A}_{\text{rad}} \right] = \frac{1}{c} \frac{\partial}{\partial T} \left[ \frac{q}{4\pi r} \frac{\mathbf{n} \times \mathbf{n} \times \mathbf{\beta}}{(1 - \mathbf{n} \cdot \mathbf{\beta})} \right]
\]

(13.15)

\[
= \frac{q}{4\pi c^2} \left[ \mathbf{n} \times (\mathbf{n} - \mathbf{\beta}) \times \mathbf{a} \right] \frac{1}{(1 - \mathbf{n} \cdot \mathbf{\beta})^2} \left( \frac{\partial}{\partial T} \right)_{\text{ret}}
\]

(13.16)

(g) We will often be interested in the frequency distribution of the radiation.

\[
\mathbf{E}(\omega, r) \equiv \int_{-\infty}^{\infty} dt \ e^{i\omega t} \mathbf{E}(t, r)
\]

(13.17a)

\[
= \frac{q}{4\pi c^2} \int_{-\infty}^{\infty} dT \ e^{i\omega(T - \mathbf{r} \cdot \mathbf{n}(T)/c)} \frac{\mathbf{n} \times (\mathbf{n} - \mathbf{\beta}) \times \mathbf{a}}{(1 - \mathbf{n} \cdot \mathbf{\beta})^2}
\]

(13.17b)

\[
= \frac{q}{4\pi c^2} \int_{-\infty}^{\infty} dT \ e^{i\omega(T - \mathbf{r} \cdot \mathbf{n}(T)/c)} \frac{\mathbf{n} \times \mathbf{n} \times \mathbf{\beta}}{(1 - \mathbf{n} \cdot \mathbf{\beta})^2}
\]

(13.17c)

We are computing the fourier transform of \(\mathbf{E}_{\text{rad}}(t, r)\) to find \(\mathbf{E}_{\text{rad}}(\omega, r)\). Changing variables to integrate over \(T\) instead of \(t\) yields Eq. (13.17b) with Eq. (13.13d). Integrating by parts using Eq. (13.15) yields Eq. (13.17c). This final form Eq. (13.17c) is often the most convenient, but sometimes it is just easier to use Eq. (13.17b) which shows explicitly the dependence on acceleration.

**Observables in the far field**

(a) The energy per time per solid angle received at the detector is

\[
\frac{dW}{dtd\Omega} = \frac{dP(t)}{d\Omega} = r^2 S \cdot n = c |r \mathbf{E}|^2
\]

(13.18)

(13.19)

This is what you want to know if you want to find out if the detector will burn up.
13.2 RELATIVISTIC LARMOUR

(b) We often want to know how much energy was radiated over a given period of acceleration, \( T_1 \ldots T_2 \). For example, how much energy was lost by the particle as it moved through one complete circle. Then we want to evaluate the energy radiated per retarded time from \( T_1 \) up to the time it completes the circle \( T_2 \)

\[
\frac{dW}{dTd\Omega} = \frac{dP(T)}{dT} = r^2 S \cdot n \frac{dt}{dT} = c |rE|^2 (1 - n \cdot \beta) 
\]

(c) We are also interested in the frequency distribution of the emitted radiation. The energy per solid angle is

\[
(2\pi) \frac{dW}{d\omega d\Omega} = \frac{c |rE(\omega, r)|^2}{2\pi} + \frac{c |rE(-\omega, r)|^2}{2\pi} = \frac{c |rE(\omega, r)|^2}{\pi} 
\]

Since the sign of \( \omega \) is without significance (for real fields such as the electromagnetic fields), we sometimes use

\[
\frac{dI}{d\omega d\Omega} = \frac{c |rE(\omega, r)|^2}{2\pi} + \frac{c |rE(-\omega, r)|^2}{2\pi} = \frac{c |rE(\omega, r)|^2}{\pi} 
\]

So that

\[
\frac{dW}{d\Omega} = \int_0^{\infty} \frac{dI}{d\omega d\Omega} 
\]

(d) The energy spectrum can be interpreted as the average number of photons per frequency per solid angle

\[
\frac{dI}{d\omega d\Omega} = \hbar \omega \frac{dN}{d\omega d\Omega} 
\]

13.2 Relativistic Larmour

(a) For a particle undergoing arbitrary relativistic motion, we evaluated the energy per retarded time per solid angle

\[
\frac{dP(T)}{d\Omega} = \frac{q^2}{16\pi^2 c^3} \frac{|n \times (n - \beta) \times a|^2}{(1 - n \cdot \beta)^5} 
\]

(b) Integrating over angles we get

\[
P(T) = \frac{dW}{dT} = \frac{q^2}{4\pi^2} \frac{2}{3} \gamma^6 \left[ a^2 \gamma^3 + \frac{a^2}{\gamma^2} \right] 
\]

where \( a_\parallel \) is the projection of \( a = d^2x/dt^2 \) along the direction of motion, and \( a_\perp \) is the component of \( a \) perpendicular to the direction of motion, i.e. for \( v \) in the z direction

\[
a = (a_\parallel, a_\perp, a_\parallel) 
\]

(c) The acceleration four vector is

\[
a^\mu = \frac{d^2x^\mu}{dt^2} 
\]

For a paraticle moving along in the z-direction, the acceleration in the particle’s locally inertial frame (i.e. the frame that is instantaneously moving with the particle) is

\[
(a^0, a^1, a^2, a^3)|_{\text{rest frame}} = (0, a_\parallel, a_\perp, a_\parallel) 
\]

While in the lab frame \( a^\mu \) is found by boosting this result. The acceleration \( a = \frac{dv}{d\tau} \) is found from this result and the definition of proper time \( d\tau = dt/\gamma \),

\[
a = (a_\parallel, a_\perp, a_\parallel) = (\gamma^2 a_\parallel, \gamma^2 a_\perp, \gamma^3 a_\parallel) 
\]

You should be able to prove this. The relativistic Larmour forumula can then be written

\[
P(T) = \frac{q^2}{4\pi^2} \frac{2}{3c^3} \langle a_\parallel a_\mu a^\mu \rangle 
\]
(d) For straight line acceleration at very large $\gamma$, we found that the radiation is emitted within a cone of order
\[ \Delta \Theta \sim 1/\gamma. \]  
(13.33)

For $\theta$ very small $\theta \sim 1/\gamma$ we found,
\[ \frac{dP(T)}{d\Omega} = \frac{2q^2 a^2}{\pi^2 c^3 \gamma^8} \frac{(\gamma \theta)^2}{(1 + (\gamma \theta)^2)^5}. \]  
(13.34)

You should feel comfortable deriving this result.