Covariant Electrodynamics

1. $\partial_{\mu} = \partial_{\frac{1}{c^2 \partial t}} \left( \begin{array}{c} \frac{1}{c} \\ \nabla \end{array} \right)$ transforms as a four vector.

There is also a contravariant component $\partial^{\mu} = \left( -\frac{1}{c} \partial_t, \frac{\nabla}{c} \right)$.

So, $\partial_{\mu} \partial^{\mu} = -\frac{1}{c^2} \frac{\partial}{\partial t} + \nabla^2 \equiv \Box \frac{\partial}{\partial x_\mu}$.

is invariant.

2. Then there is the continuity Eqn.

$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \Rightarrow \frac{1}{c} \frac{\partial}{\partial t} \left( \rho \right) + \nabla \cdot \mathbf{J} = 0$

So take $\mathbf{J}^{\mu} = \left( \rho, \mathbf{J} \right)$, as a four vector,

$\partial_{\mu} \mathbf{J}^{\mu} = 0$

3. Then the equations for the gauge potential

$-\Box \psi = \mathbf{J}^0 / c$

$-\Box \mathbf{A} = \mathbf{J} / c$

Together with the lorentz gauge condition:

$\frac{1}{c} \frac{\partial}{\partial t} \left( \rho \right) + \nabla \cdot \mathbf{A} = 0$.
So in order to have a lorentz invariant theory take \((\Psi, \vec{A})\) to be a four vector

\[ A^m = (\Psi, \vec{A}) \]

Then the wave eqn becomes

\[ -iDA^m = \mathbf{J}^m / c \quad \text{and} \quad 2_\mu A^\mu = 0 \]

(4) Now the fields

\[
\begin{align*}
\vec{E} &= -i2A - \vec{\nabla}\Psi \\
\vec{B} &= \nabla \times \vec{A}
\end{align*}
\]

These are combined into a rank 2 antisymmetric tensor

Where

\[
F^{\alpha\beta} = \left( \begin{array}{ccc}
0 & E^x & E^y \\
-E^x & 0 & B^z \\
-E^y & -B^z & 0 \\
-E^z & B^y & -B^x \\
\end{array} \right) \equiv \partial^x A^\beta - \partial^\beta A^x
\]

We can see this
\[ F^{0i} = E^{i} = -\left( \frac{\partial A^{i}}{\partial t} - \frac{\partial \psi}{\partial x^{i}} \right) \Rightarrow c \frac{\partial t}{\partial t} = \frac{\partial x^{i}}{\partial x^{i}}. \]

\[ B_{k} = (\nabla \times A)_{k} \]

\[ F^{ij} = \varepsilon^{ijk} B_{k} = \varepsilon^{ijk} \varepsilon_{klm} \frac{\partial A^{m}}{\partial x^{k}} \cdot (\delta^{i}_{j} \delta^{l}_{m} - \delta^{i}_{m} \delta^{j}_{k}) \frac{\partial A^{m}}{\partial x^{k}} = \frac{\partial A^{j}}{\partial x^{i}} - \frac{\partial A^{i}}{\partial x^{j}} \]

So, \( F^{\alpha \beta} = \frac{\partial \alpha^{A}}{\partial x^{\beta}} \) transforms as a second rank tensor in the following way

\[ F^{\mu \nu} = L^{\mu}_{\alpha} L^{\nu}_{\beta} F^{\alpha \beta} \]

Excercise,

Show that

\[ F^{i} = F^{0i} = -F^{i0} = F^{i0} = -F^{i} = F^{0i} \]
Now the Eom (Part I) \[ \nabla \cdot E = \rho \Rightarrow -\partial_t F^{\alpha 0} = J^\alpha / c \]

and \[ -\frac{1}{c} \partial_t E + \nabla \times B = J \Rightarrow -\left( \frac{\partial F^{\alpha i}}{\partial x^0} + \frac{\partial F^{\xi i}}{\partial x^\xi} \right) = J^i / c \]

So \[ -\partial_\alpha F^{\alpha \beta} = J^\beta / c \]

Exercise:

Starting from \[ -\partial_\alpha F^{\alpha \beta} = J^\beta / c \] and the definition of \( F^{\alpha \beta} \) derive:

\[ -\nabla A^\beta = J^\beta / c \]

Solution \[ = F^{\alpha \beta} \]

\[ -\partial_\alpha (\delta^\alpha A^\beta - \delta^\beta A^\alpha) = -\partial_\alpha \partial^\alpha A^\beta + \partial^\beta (\partial_\alpha A^\alpha) = J^\beta \]

Lorentz Gauge \( \partial_\alpha A^\alpha = 0 \), so

\[ -\partial_\alpha \partial^\alpha A^\beta = 0 \quad \text{or} \quad \nabla A^\beta = J^\beta \]
6) The Remaining Maxwell Eqs

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \]

Comparison with the first two eqs in absence of currents gives

\[ \nabla \cdot \mathbf{E} = 0 \]

So the second two Maxwell eqs involve the replacement (duality).

\[ \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = 0 \]

\[ \mathbf{E} \rightarrow \mathbf{B} \text{ and } \mathbf{B} \rightarrow -\mathbf{E}. \]

Thus define the dual tensor

\[ \tilde{F}^{\mu \nu} = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & -E^z & E^y \\ -B^y & E^z & 0 & -E^x \\ -B^z & -E^y & E^x & 0 \end{pmatrix} \]

So

\[ \partial_{\nu} \tilde{F}^{\mu \nu} = 0 \]
The dual tensor can be defined from $F_{\mu\nu}$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$  

this implements the replacement $E \to B$, $B \to -E$

Here

$$\varepsilon^{\mu\nu\alpha\beta} = \begin{cases} 
+1 & \text{for even perms of } 0,1,2,3 \\
-1 & \text{for odd perms of } 0,1,2,3 \\
0 & \text{otherwise}
\end{cases}$$

Expressing in terms of antisymmetric in $\mu\alpha\beta$

$$\partial_\mu \tilde{F}^{\mu\nu} = -\frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \partial_\mu F_{\alpha\beta} = 0$$

This can be written as the Bianchi Identity

$$\partial_{[\mu} F^{\nu\rho]} = 0$$

or

$$\partial_\mu F_{\nu\rho\sigma} - \partial_\nu F_{\mu\rho\sigma} + \partial_\rho F_{\mu\nu\sigma} = 0$$

where $F_{\mu\nu\rho\sigma}$ stands for the antisymmetric combo.

Examples:

$$T_{[m_1, m_2]} = \frac{1}{2!} (T_{m_1 m_2} - T_{m_2 m_1})$$

like a $2 \times 2$ determinant

$$T_{[m_1, m_2, m_3]} = \frac{1}{3!} \left[ (T_{m_1 m_2 m_3} - T_{m_1 m_3 m_2} - T_{m_2 m_3 m_1}) + (T_{m_3 m_1 m_2} - T_{m_3 m_2 m_1}) \right]$$

like a $3 \times 3$ determinant.
Exercise

Show that is \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) then the second two Maxwell eqs are automatically satisfied.

Solution

\[
\partial_\mu F^{\mu\nu} = \partial_\mu \left( \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \left( \partial_\rho A_\sigma - \partial_\sigma A_\rho \right) \right)
\]

\[
= -\frac{1}{2} \varepsilon^{\nu\mu\rho\sigma} \left( \partial_\rho A_\sigma - \partial_\sigma A_\rho \right) = 0
\]

But \( \partial_\mu \partial_\nu A_\rho = \partial_\nu \partial_\mu A_\rho \) and \( \varepsilon^{\nu\mu\rho\sigma} = -\varepsilon^{\nu\sigma\rho\mu} \)

Symmetric \quad \text{antisymmetric}

And the contraction of antisymm and a symmetric tensor gives zero.
Last Time

- Finished by discussing the stress tensor:

\[ \Theta^{\mu\nu}_{\text{Tot}} = \begin{pmatrix} u_{\text{Tot}} & \frac{\dot{S}_{\text{Tot}}}{c} \\ \frac{c \dot{g}_{\text{Tot}}}{c} & T^{i\dot{j}} \end{pmatrix} \quad \text{with} \quad \partial_{\mu} \Theta^{\mu\nu}_{\text{Tot}} = 0 \]

E-Conservation

- 0-component

\[ \Theta^{00}_{\text{Tot}} = \text{energy density} = u_{\text{Tot}} \] \[ \partial_{\mu} \Theta^{00}_{\text{Tot}} = 0 \]

\[ \Theta^{0i}_{\text{Tot}} = \text{energy flux} = \frac{\dot{S}}{c} = \dot{g}_{i} \]

Mom-Conservation

- i-th component

\[ \Theta^{0i}_{\text{Tot}} = \text{momentum density} = \dot{g}_{i} = \frac{\dot{S}}{c} \] \[ \partial_{\mu} \Theta^{0i}_{\text{Tot}} = 0 \]

\[ \Theta^{ij}_{\text{Tot}} = \text{stress force/area} = T^{ij} \]

If I have a mechanical system (like a fluid), with currents then the E&M fields will push and pull the system:

\[ \partial_{\mu} \Theta^{\mu\nu}_{\text{mech}} = F^{\nu} \cdot \frac{J P}{c} \]

\[ \partial_{\mu} \Theta^{00}_{\text{mech}} = \frac{E}{c} \]

\[ \partial_{\mu} \Theta^{0i}_{\text{mech}} = \rho \dot{E}^{i} + \left( \frac{J \times B}{c} \right) \]

And thus mechanical energy and momentum won't be conserved.
The electromagnetic force must be the divergence of something:

\[ F^\mu \frac{\partial J^\rho}{\partial x^\mu} = - \partial_x \Theta_{em}^{\mu \nu} \]

Homework: show using \(-\partial_x F^\mu = J^\mu / c\) that

\[ \Theta_{em}^{\mu \nu} = F^\alpha_\mu F_\nu^\alpha + g^{\mu \nu} \left( \frac{-1}{4} F^2 \right) \]

(see below)

Then

\[ \partial_\mu \Theta_{mech}^{\mu \nu} = - \partial_x \Theta_{em}^{\mu \nu} \]

or

\[ \partial_\mu (\Theta_{mech}^{\mu \nu} + \Theta_{em}^{\mu \nu}) = 0 \]

and thus the combined mechanical + electromagnetic energy and momentum will be conserved.

\[ \Theta_{em}^{\mu \nu} = \begin{pmatrix} \frac{1}{2}(E^2 + B^2) & \vec{E} \times \vec{B} \\ \vec{E} \times \vec{B} & \vec{E} \vec{B} + \frac{1}{2} \epsilon^{ijk} E^i E^j B^k \end{pmatrix} = \begin{pmatrix} u_{em} & \frac{1}{c} \omega_{em} \\ \vec{g}_{em} & \vec{t}_{em} \end{pmatrix} \]