Problem 1. One liners

(a) Starting from the Maxwell equations for $F^{\mu\nu}$ and the definition of $F^{\mu\nu}$, derive the wave equation $-\Box A^{\mu} = J^{\mu}/c$.

(b) Starting from the maxwell equations for $F^{\mu\nu}$ in covariant form, show that we must have $\partial_{\mu} J^{\mu} = 0$ for consistency.

(c) (This is two lines) Show that the energy conservation and force laws

$$\frac{dE_{\boldsymbol{p}}}{dt} = q\boldsymbol{E} \cdot \boldsymbol{v}_p \tag{1}$$

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + \frac{\boldsymbol{v_p}}{c} \times \boldsymbol{B}) \tag{2}$$

can be written covariantly

$$\frac{dP^{\mu}}{d\tau} = F^{\mu\nu} u_{\nu}/c \tag{3}$$

Note that $E_{\boldsymbol{p}}$ (the energy of the particle) is different from $\boldsymbol{E}$ the electric field.

(d) From Eq. (3) show that $P_{\mu}P^{\mu}$ is constant in time.

(e) Show that $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ is invariant under the gauge transform

$$A_{\mu} \to A_{\mu} + \partial_{\mu}\Lambda(X) \tag{4}$$

where Λ is an arbitrary function of $X = (t, \boldsymbol{r})$.

(f) Given $F^{\mu\nu}$ the only two Lorentz invariant quantities are $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu}$. Evaluate these two invariants in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$[1]

[1] answers: $2(B^2 - E^2)$ and $-4E \cdot B$