Short in Class Exercise - Invariants of $F^{\mu\nu}$

Given $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$, the only two invariants quadratic in the fields are

$$F_{\mu\nu} F^{\mu\nu} \quad \text{and} \quad F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Evaluate these in terms of $E$ and $B$:

Ans:

$$F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2) = 2F_0 F_0 + F_2 F_2$$

$$F_{\mu\nu} \tilde{F}^{\mu\nu} = 4B \cdot E$$

Solution:

$$\begin{bmatrix}
0 & B^x & B^y & B^z \\
-B^x & 0 & -E^z & E^y \\
-B^y & E^z & 0 & -E^x \\
-B^z & -E^y & E^x & 0
\end{bmatrix} \quad \begin{bmatrix}
0 & -E^x & -E^y & -E^z \\
+E^x & 0 & B^z & -B^y \\
-E^y & B^z & 0 & B^x \\
-E^z & -B^y & -B^x & 0
\end{bmatrix}$$

Just multiply it out.
Transformation of Fields

\[ F^{\mu\nu}(x) = L^\mu \alpha L^\nu \beta F^{\alpha\beta} \]

For a boost in the \( x \)-direction

\[ L^\mu \gamma = \begin{pmatrix} \gamma & -\gamma \beta \\ \gamma \beta & \gamma \end{pmatrix} \]

* Then first look in parallel \( x \) direction

\[ E_x = E^x = F_0^1 = L^0 \alpha L^1 \beta F^{0\beta} \]

\[ = L^0 L^1 F_0^1 + L^0 L^0 F_{10} \]

\[ = (\gamma^2 - \gamma^2 \beta^2) F_0^1 \]

\[ = E_x \]

* Similarly in \( z \) direction

\[ E_z = E^y = F_0^2 = L^0 L^2 \beta F^{0\beta} \]

\[ = L^0 L^2 F_0^2 + L^0 L^0 F_{12} \]

\[ = \gamma E^2 + \gamma \beta \beta^3 \]
So we find
\[ \begin{align*}
\vec{E}'' &= \gamma \vec{E}' \\
\vec{B}'' &= \vec{B}' \\
\vec{E}'_\perp &= \gamma \vec{E}'_\perp + \vec{\beta} \times \vec{B}'_\perp \\
\vec{B}'_\perp &= \gamma \vec{B}'_\perp - \vec{\beta} \times \vec{E}'_\perp
\end{align*} \]

1) Looks like coordinate transforms, but it is the transverse pieces which get boosted.

2) The transformation of \( \vec{B} \) is the dual of \( \vec{E} \)
   \[ E \rightarrow \vec{B}, \quad \vec{B} \rightarrow -E \]

3) Very often one has an electrostatic field \( (\vec{E} = \vec{E}^0, \quad \vec{B} = 0) \) and we want to know \( \vec{B} \) in the new frame:
   \[ \begin{align*}
   \vec{B} &= -\vec{\beta} \times \gamma \vec{E}_\perp \\
   &= -\vec{\beta} \times \vec{E}_\perp \\
   &= -\vec{\beta} \times \vec{E}_\perp \quad \parallel \text{components } \vec{\beta} \times \vec{E} = 0 \\
   \end{align*} \]
   So we can drop ± symbol.
Fields of a moving Particle

- Particle at Rest:

\[ F^{\mu \nu} = \text{coulomb field} \]

\[ F^{\mu \nu} (x) = ? \]

- Person sees a particle approach:

\[ b \]

- The boost to the person frame from the particle frame is:

\[ \mathbf{L}^{\mu \nu} = \begin{pmatrix} \gamma + \gamma \beta & \gamma \beta & \gamma & 0 \\ \gamma \beta & \gamma & 0 & 0 \\ \gamma & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ \hat{\beta} = \text{velocity of particle as seen by person} = -\hat{\beta}_{\text{frame}} \]
\[ E_1(x) = \frac{q}{4\pi(r^2)} x^i = E_{\parallel}(x) \Rightarrow E_{\parallel} = \frac{q}{4\pi r^3} \]

\[ E_2(x) = \frac{q}{4\pi(r^2)} x^i = (E_{\perp}) \Rightarrow E_{\perp} = \frac{q}{4\pi r^3} \]

So under boost:

\[ X^\mu = L^\mu \nu \ x^\nu \]

\[ (L^{-1})^\nu \ x^\mu = x^\nu \]

\[ x^1 = \gamma (x^1 - vt) \]

\[ x^2 = \frac{x_2}{b} \]

\[ x^3 = \frac{x_3}{b} \]

\[ b = (x_2, x_3) \]

So using our rules:

\[ E_{\parallel}(x) = E_{\parallel}(x) \]

\[ E_{\parallel} = \frac{q}{4\pi} \frac{\gamma (x^1 - vt)}{[\gamma^2(x^1 - vt)^2 + b^2]^{3/2}} \]

\[ E_{\perp}(x) = \gamma E_{\perp}(x) = \frac{q}{4\pi} \frac{\gamma b}{[\gamma^2(x^1 - vt)^2 + b^2]^{3/2}} \]

\[ \beta = \beta_{\text{frame}} \longleftrightarrow \text{note sign is opposite because } \beta \text{ is velocity of particle not the frame} \]
So the picture at large $r$ is

$$\Delta x = (x^1 - vt) \sim \frac{b}{r}$$

(Side View)

Comment:

- The field is appreciable when $x^1 - vt$ is of order $b/r$
- Plotting the field strength one sees that the transverse field is much larger

$$E_{\perp}^\text{max} = \frac{q \gamma}{4\pi b^2}$$

$$\Delta x \sim \frac{b}{r}$$

$$E_{\parallel}^\text{max} \sim \frac{q}{4\pi b^2}$$

i.e., the transverse fields are enhanced by $\gamma$ while the longitudinal fields remain finite.
Find that the transverse fields give a finite impulse, but the longitudinal kick is zero (Homework).

\* For \( \beta \rightarrow 1 \) the \( \vec{E} \) and \( \vec{B} \) fields act like plane waves, i.e.

\[ |\vec{E}| \approx |\vec{B}| \quad \vec{E} \times \vec{B} = \vec{z} \]

The head-on view, \( \vec{B} \) is perpendicular to \( \vec{E} \)

\[ \vec{S} = c (\vec{E} \times \vec{B}) \text{ points out of the page} \]
The fact that \( \hat{B} \) is perpendicular to \( \hat{E} \) could have been anticipated

\[
F_{\mu \nu} F^{\mu \nu} = -4 \hat{E} \cdot \hat{B}
\]

is Lorentz invariant. In the frame of the particle, \( \hat{B} \) is zero so in the particle frame:

\[
F_{\mu \nu} \cdot F^{\mu \nu} = 0
\]

So in any other frame we must have \( \hat{E} \cdot \hat{B} = 0 \)
Moving Media

- For stationary conductors

\[ \vec{j} = \sigma \vec{E} + \frac{\varepsilon \varepsilon_0}{\mu_0} \nabla \times \vec{B} \]

- It is reasonably clear that the appropriate generalization is for moving conductors with velocity \( \vec{v} \) is:

\[ \vec{j} = \sigma \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) \]

to first order in \( \vec{v}/c \). Let's see it

\[ \begin{align*}
\text{Sample} & \quad \vec{E}, \vec{B} \quad \rightarrow \quad \vec{V} \\
\text{current in electric field} & \quad \text{in sample frame} \\
\text{sample frame} & \quad \vec{j} = \sigma \vec{E} = \sigma \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) \quad \text{from transformation} \\
\end{align*} \]

Now we have the properties of \( \vec{E} + \vec{B} \) to compute the current in lab frame no net charge

\[ \left( \begin{array}{c}
\frac{c}{\gamma} \\
\vec{j}_{\text{lab}}
\end{array} \right) = \left( \begin{array}{cc}
\gamma & \gamma \beta \\
\beta & \gamma
\end{array} \right) \left( \begin{array}{c}
\frac{c}{\gamma} \\
\vec{j}
\end{array} \right) = \left( \begin{array}{cc}
1 & \beta \\
\beta & 1
\end{array} \right) \left( \begin{array}{c}
0 \\
\frac{\vec{j}}{\gamma}
\end{array} \right) \]

So

\[ \vec{j}_{\text{lab}} \approx \frac{\vec{j}}{\gamma} = \sigma \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) \]
Example

\[ \vec{B} = B_0 \hat{x} \]

A uniform conducting cylinder rotated with angular velocity \( \omega \). Determine the torque required to maintain the motion. Cylinder has conductivity \( \sigma \) and radius \( a \).

Solution

\[ \frac{1}{\sigma} \frac{\partial}{\partial t} (\vec{V} \times \vec{B}) = \frac{1}{c} \sigma \omega \hat{r} \left( -\sin \phi \hat{x} + \cos \phi \hat{y} \right) \times B_0 \hat{x} \]

\[ \frac{1}{\sigma} \frac{\partial}{\partial t} \vec{J} = \frac{B_0 \sigma \omega r \cos \phi}{c} (-\hat{z}) \]

So the forces are

Top view

The remaining steps are easy:

\[ \vec{L}_{em} = L \int d^2r \; \vec{r} \times \left( \frac{1}{c} \frac{\partial}{\partial t} \vec{B} \right) = \int d^2r \: \frac{1}{c} \left( \vec{F} \cdot \vec{B} \right) - \left( \vec{F} \times \frac{\vec{B}}{c} \right) \frac{1}{c} \]

\[ = L \int_0^a \int_0^{2\pi} r dr d\phi \; (-\hat{z}) (B_0 \sigma \omega r \cos \phi / c^2) (B_0 \hat{r} \cos \phi) \]
So then

\[
\frac{T_{em}}{L} = \text{torque per length}
\]

\[
= \left( B_0^2 \frac{a^4}{4} \sigma \omega \cdot \Pi \right) (-\hat{z})
\]

Units:

\[ B_0^2 = \frac{N}{m^2} \quad S_0 \]

\[ a^4 = m^4 \]

\[ \omega \sigma = \frac{1}{s^2} \]

\[
\begin{bmatrix}
\frac{T_{em}}{L}
\end{bmatrix} = \frac{N}{m^2} \frac{m^4}{s^2} \frac{1}{s^2} \frac{1}{s^2} = N\sqrt{\frac{s^2}{s^2}}
\]

\[ C^2 = \frac{m^2}{s^2} \]