Short in Class Excercise - Invariants of $F^{\mu\nu}$

Given $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ the only two invariants quadratic in the fields are

$$F_{\mu\nu}F^{\mu\nu} \quad \text{and} \quad F_{\mu\nu}\tilde{F}^{\mu\nu}$$

Evaluate these in terms of E and B:

Ans:

$$F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2) = \overbrace{2F_{0i}F^{0i}}^{-2E^2} \overset{+}{+} \overbrace{F_{ij}F^{ij}}^{2B^2}$$

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\,B\cdot E$$

Solution:

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & E^z & -E^y \\ -B^y & -E^z & 0 & E^x \\ -B^z & E^y & -E^x & 0 \end{pmatrix} \qquad F_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & -E^x & -E^y & -E^z \\ \hline +E^x & 0 & B^z & -B^y \\ E^y & -B^z & 0 & B^x \\ E^z & B^y & -B^x & 0 \end{array}\right)$$

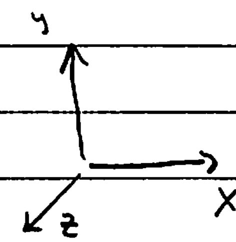
Just multiply it out

Transformation of Fields

$$\underline{F}^{\mu\nu}(x) = L^{\mu}{}_{\alpha} L^{\nu}{}_{\beta} F^{\alpha\beta}$$

For a boost in the x-direction

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$



- Then first look in parallel x_1 direction

$$\underline{E}_{\parallel} = \underline{E}^{x} = \underline{F}^{01} = L^{0}{}_{\alpha} L^{1}{}_{\beta} F^{\alpha\beta}$$

$$= L^{0}{}_{0} L^{1}{}_{1} F^{01} + L^{0}{}_{1} L^{1}{}_{0} F^{10}$$

$$= (\gamma^2 - \gamma^2\beta^2) F^{01}$$

$$= E_{\parallel}$$

- Similarly in $\perp$ direction

$$\underline{E}_{\perp} = \underline{E}^{y} = \underline{F}^{02} = L^{0}{}_{\alpha} L^{2}{}_{\beta} F^{\alpha\beta}$$

$$= L^{0}{}_{0} L^{2}{}_{2} F^{02} + L^{0}{}_{1} L^{2}{}_{2} F^{12}$$

$$\underline{E}^{2} = \gamma E^2 + \gamma\beta B^3$$

So we find

$$\boxed{\begin{aligned} \underline{\vec{E}}_{\parallel} &= \vec{E}_{\parallel} & \qquad \underline{\vec{B}}_{\parallel} &= \vec{B}_{\parallel} \\ \underline{\vec{E}}_{\perp} &= \gamma \vec{E}_{\perp} + \gamma \vec{\beta} \times \vec{B}_{\perp} & \qquad \underline{\vec{B}}_{\perp} &= \gamma \vec{B}_{\perp} - \gamma \vec{\beta} \times \vec{E}_{\perp} \end{aligned}}$$

① Looks like coordinate transforms, but it is the transverse pieces which get boosted.

② The transformation of $\vec{B}$ is the dual of $\vec{E}$
$\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$

③ Very often one has an electrostatic field ($\vec{E} = \vec{E}^{(0)}$, $\vec{B} = 0$), and we want to know $\underline{\vec{B}}$ in the new frame:

$$\underline{\vec{B}} = -\vec{\beta} \times \gamma \vec{E}_{\perp}$$

$\underline{\vec{E}}_{\perp} = \gamma \vec{E}_{\perp}$

$$= -\vec{\beta} \times \underline{\vec{E}}_{\perp}$$

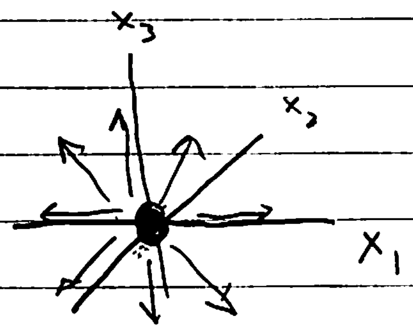
$\parallel$ components $\times \vec{\beta} = 0$
So we can drop
$\perp$ symbol

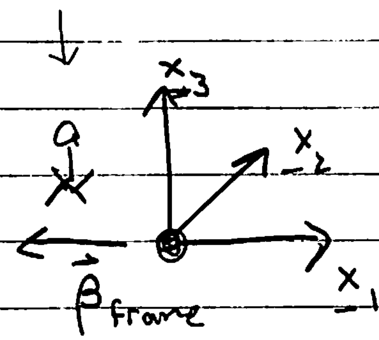
$$= -\vec{\beta} \times \underline{\vec{E}}$$

Fields of A moving Particle

- Particle at Rest:

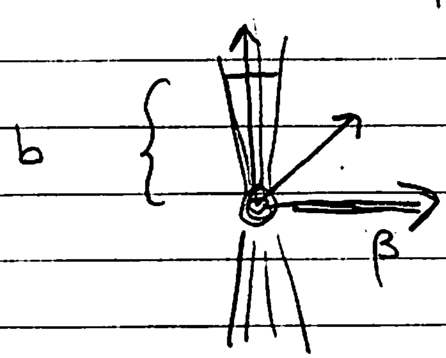
person approaching

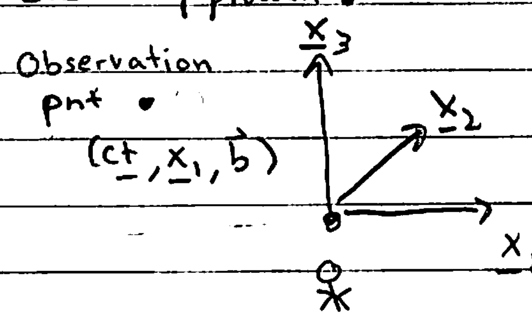




$F^{\mu\nu}$ = coulomb field $\qquad\qquad$ $\underline{F}^{\mu\nu}(\underline{x}) = ?$

- Person sees a particle approach:





- The boost to the person frame from the partice frame is:

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & +\gamma\beta & & \\ +\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$\vec{\beta}$ = velocity of particle as seen by person

$= -\vec{\beta}_{frame}$

Now

$$E_1(x) = \frac{q}{4\pi(x_i x^i)^{3/2}}\, x_1 = E_{\parallel}(x) \quad \Rightarrow \quad \vec{E}_{\parallel} = \frac{q\,\vec{z}}{4\pi r^3}$$

$$E_2(x) = \frac{q}{4\pi(x_i x^i)^{3/2}}\, x_2 = (\vec{E}_{\perp})_2 \quad \Rightarrow \quad \vec{E}_{\perp} = \frac{q\,\vec{b}}{4\pi r^3}$$

So under boost:

$$\left.\begin{array}{l} \underline{x}^\mu = L^\mu{}_\nu\, x^\nu \\ (L^{-1})^\nu{}_\mu\, \underline{x}^\mu = x^\nu \end{array}\right\} \qquad \begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & \\ -\gamma\beta & \gamma & \\ & & 1 \end{pmatrix} \begin{pmatrix} c\underline{t} \\ \underline{x}^1 \\ \underline{b} \end{pmatrix}$$

$$x^1 = \gamma(\underline{x}^1 - v\underline{t})$$

$$\begin{matrix} x^2 \\ x^3 \end{matrix} = \begin{matrix} \underline{x}_2 \\ \underline{x}_3 \end{matrix} \qquad \vec{b} = (x_2, x_3)$$

So using our rules:

$$E_{\parallel}(\underline{x}) = E_{\parallel}(x)$$

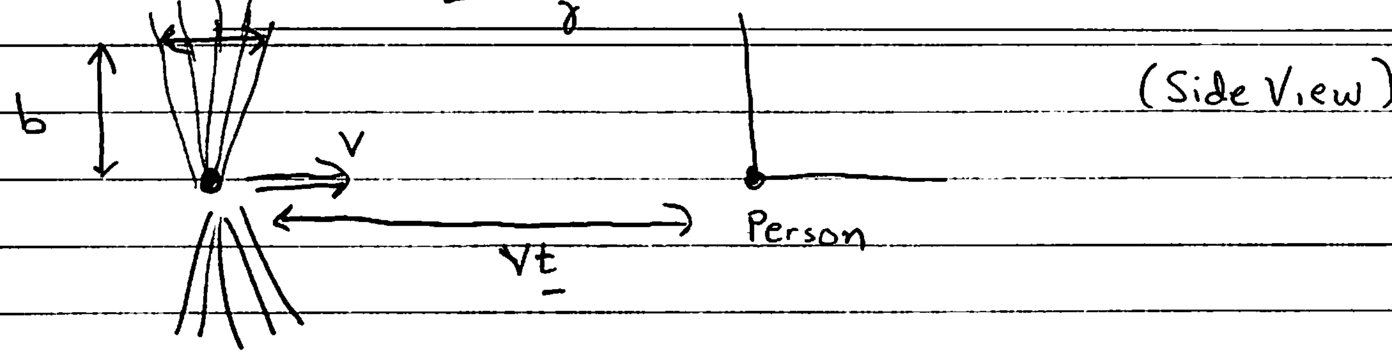
$$E_{\parallel} = \frac{q\gamma(\underline{x}^1 - v\underline{t})/4\pi}{\left[\gamma^2(\underline{x}^1 - v\underline{t})^2 + b^2\right]^{3/2}}$$

$$E_{\perp}(\underline{x}) = \gamma E_{\perp}(x) = \frac{q\gamma\,\vec{b}/4\pi}{\left[\gamma^2(\underline{x}^1 - v\underline{t})^2 + b^2\right]^{3/2}}$$

$\vec{\underline{B}} = \vec{\beta} \times \vec{\underline{E}}$ ← note sign is opposite because $\vec{\beta}$ is velocity of particle not the frame $= \vec{\beta}_{\text{frame}}$

So the picture at Large γ is

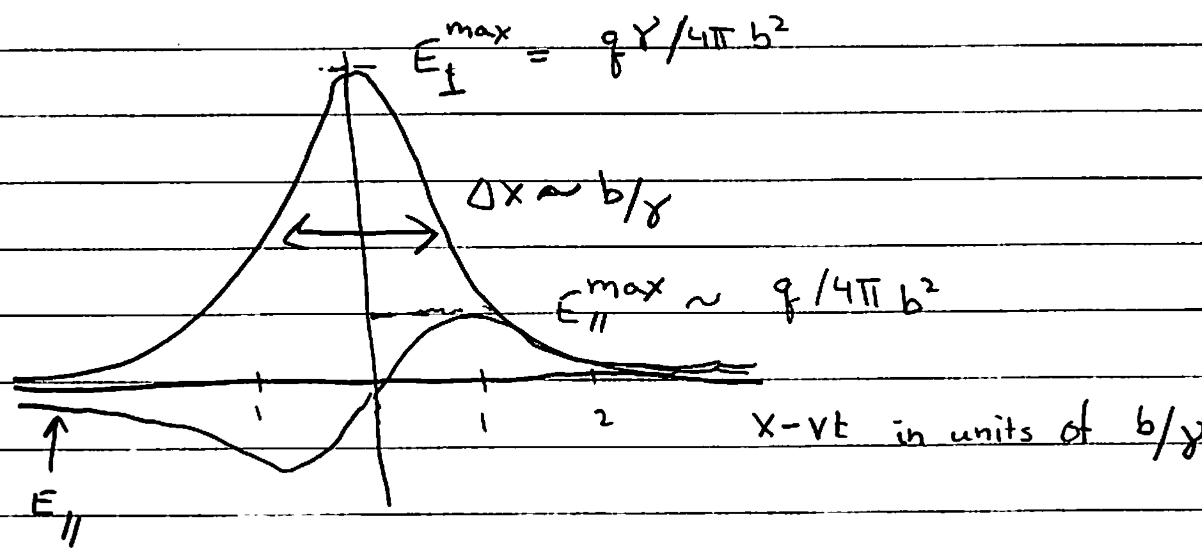
$$\Delta x = (\underline{x}' - v\underline{t}) \sim \frac{b}{\gamma}$$



Comments

- The field is appreciable when $\underset{\text{longitudinal dist from particl}}{\underline{x}_1 - v\underline{t}}$ is of order b/γ

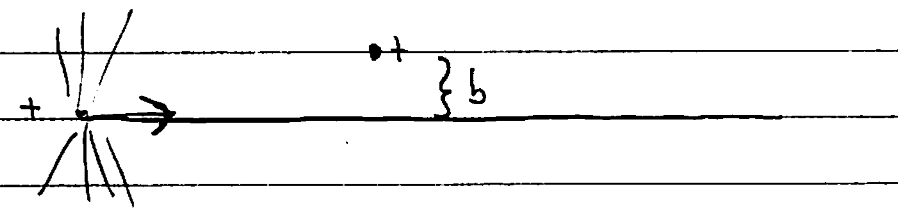
- Plotting the field strengths one sees that the transverse field is much larger



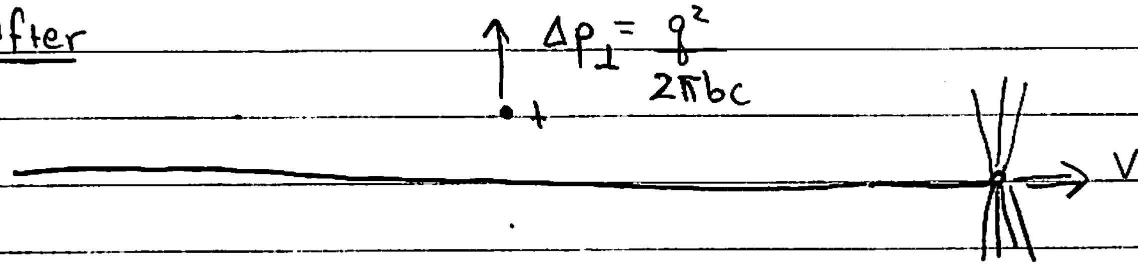
i.e. the transverse fields are enhanced by γ while the longitudinal fields remain finite

Find that the transverse fields give a finite impulse, but the longitudinal kick is zero (Homework).

Before



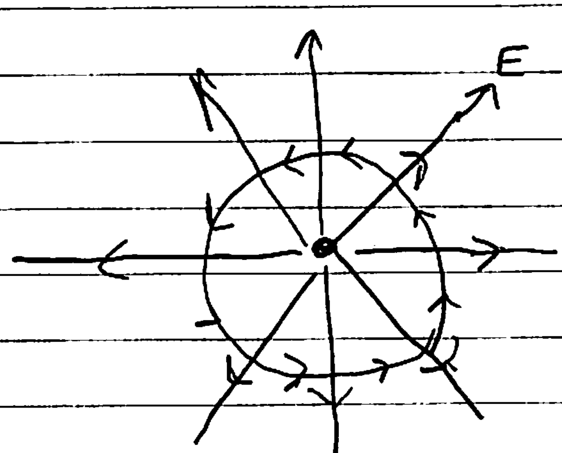
After



- For $\beta \to 1$ the $\vec{E}$ and $\vec{B}$ fields act like plane waves, ie

$$|\vec{E}| \simeq |\vec{B}| \qquad \hat{E} \times \hat{B} = \hat{z}$$

The head on view, $\vec{B}$ is perpendicular to $\vec{E}$



$\vec{S} = c\,(\vec{E} \times \vec{B})$ points out of the page

- The fact that $\vec{B}$ is perpendicular to $\vec{E}$ could have been anticipated

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\vec{E}\cdot\vec{B}$$

is Lorentz invariant. In the frame of the particle, $\vec{B}$ is zero so in the particle frame:

$$F_{\mu\nu}\cdot\tilde{F}^{\mu\nu} = 0 \quad =$$

So in any other frame we must have $\vec{E}\cdot\vec{B} = 0$

Moving Media

- For stationary conductors

$$\vec{j} = \sigma \vec{E} + \cancel{c \chi_m^B \nabla \times B}$$

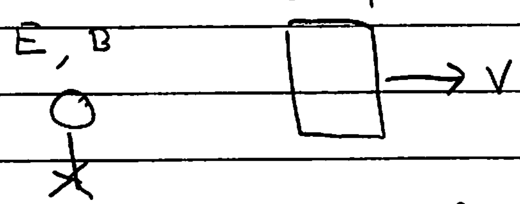
small

- It is reasonably clear that the appropriate generalization is for moving conductors with velocity $\vec{v}$ is:

$$\vec{j} = \sigma \left(E + \frac{\vec{v}}{c} \times \vec{B} \right)$$

to first order in $\vec{v}/c$. Lets see it

sample



current in sample frame

~~electric field~~ in sample frame

$$\vec{j} = \sigma \underline{E} \simeq \sigma \left(E + \frac{v}{c} \times B \right)$$

from transformation properties of E & B

Now we have to compute the current in lab frame

no net charge in sample

$$\begin{pmatrix} c\rho \\ j \end{pmatrix}_{Lab} = \begin{pmatrix} \gamma & +\gamma\beta \\ +\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix} \simeq \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \vec{j} \end{pmatrix}$$

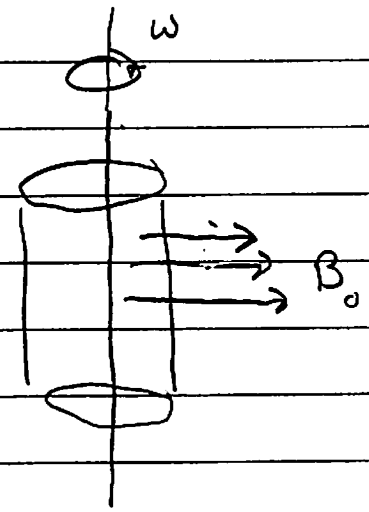
~~Current in lab~~

sample Current

So

$$\boxed{\vec{j}_{Lab} \simeq \vec{j} \simeq \sigma \left(E + v/c \times B \right)}$$

Example



$$\vec{B} = B_0 \hat{x}$$

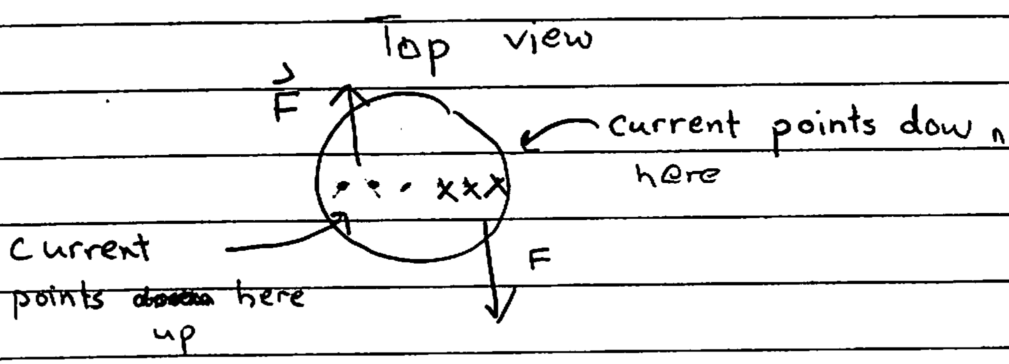
A uniform conducting cylinder rotated with angular velocity ω. Determine the torque required to maintain the motion. Cylinder has conductivity σ and radius a

Solution

$$\vec{j} = \sigma \frac{\vec{v}}{c} \times \vec{B} = \overbrace{\sigma \omega r \left(-\sin\phi \, \hat{x} + \cos\phi \, \hat{y}\right)}^{\vec{v}} \times B_0 \hat{x}$$

$$\vec{j} = \frac{B_0}{c} \sigma \omega r \cos\phi \, (-\hat{z})$$

So the forces are



The remaining steps are easy:

$$\vec{\tau}_{em} = \overset{\text{length}}{L} \int d^2 r \; \vec{r} \times \left(\frac{\vec{j}}{c} \times \vec{B}\right) = \int d^2 \vec{r} \; \frac{\vec{j}}{c} (\vec{r} \cdot \vec{B}) - \cancelto{0}{\left(\vec{r} \cdot \frac{\vec{j}}{c}\right) \vec{B}}$$

$$= L \int_0^a r \, dr \int_0^{2\pi} d\phi \; (-\hat{z}) \left(B_0 \sigma \omega r \cos\phi / c^2\right) \left(B_0 r \cos\phi\right)$$

So then

$$\frac{\vec{\tau}_{em}}{L} = \text{torque per length}$$

$$= \left(B_0^2 \frac{a^4}{4} \frac{\sigma \omega}{c^2} \cdot \pi \right) (-\hat{z})$$

Units:

$B_0^2 = N/m^2$

$a^4 = m^4$

$\omega \sigma = \frac{1}{s^2}$

$c^2 = \frac{m^2}{s^2}$

So

$$\left[\frac{\tau_{em}}{L} \right] = \frac{N}{m^2} \; m^4 \; \frac{1}{s^2} \; \frac{1}{\frac{m^2}{s^2}} = N \checkmark$$