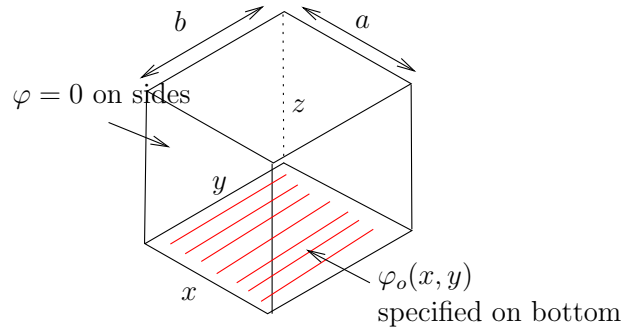


D Separation of Variables

D.1 Cartesian coordinates



(a) Laplacian

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi = 0 \quad (\text{D.1})$$

(b) Eigen functions along boundary vanishing at $x = 0$ and $x = a$ and $y = 0$ and $y = b$

$$\psi_{nm}(x, y) = \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \quad n = 1 \dots \infty \quad m = 1 \dots \infty$$

(c) Orthogonality

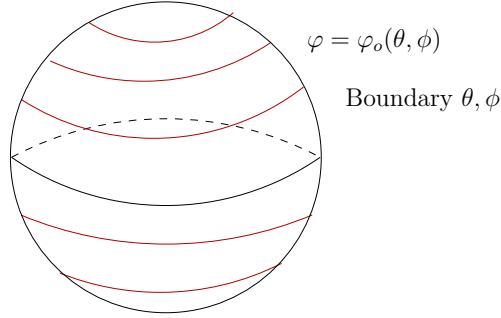
$$\int_0^a dx \int_0^b dy \psi_{nm} \psi_{n'm'} = \left(\frac{a}{2}\right) \left(\frac{b}{2}\right) \delta_{nn'} \delta_{mm'}$$

(d) Solution

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [A_{nm} e^{-\gamma_{nm} z} + B_{nm} e^{+\gamma_{nm} z}] \psi_{nm}(x, y) \quad (\text{D.2})$$

where $\gamma_{nm} = \sqrt{(n\pi/a)^2 + (m\pi/b)^2}$

D.2 Spherical coordinates



(a) Laplacian

$$\left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \Phi = 0 \quad (\text{D.3})$$

(b) Eigen functions along boundary θ, ϕ , regular at $\theta = 0$ and π , 2π periodic in ϕ

$$\psi_{\ell m}(\theta, \phi) = Y_{\ell m}(\theta, \phi) \quad \ell = 0 \dots \infty \quad m = -\ell \dots \ell$$

(c) Orthogonality:

$$\int d\Omega Y_{\ell m}^*(\theta, \phi) Y_{\ell' m'}(\theta, \phi) = \delta_{\ell \ell'} \delta_{m m'}$$

(d) Solution

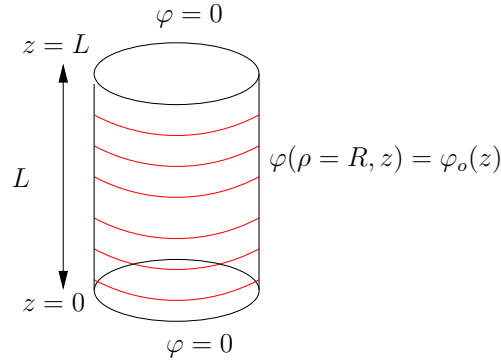
$$\Phi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}} \right] Y_{\ell m} \quad (\text{D.4})$$

(e) When there is no azimuthal dependence things simplify to

$$\Phi = \sum_{\ell=0}^{\infty} \left[A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}} \right] P_{\ell}(\cos \theta) \quad (\text{D.5})$$

where $P_{\ell}(\cos \theta)$ is the legendre polynomial, which up to a normalization is $Y_{\ell 0}(\theta, \phi)$, satisfying the orthogonality

$$\int_{-1}^1 d(\cos \theta) P_{\ell}(\cos \theta) P_{\ell'}(\cos \theta) = \frac{2}{2\ell + 1} \delta_{\ell \ell'}$$

D.3 Cylindrical Boundary: z, ϕ are the boundary.

(a) Laplacian:

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right] \Phi = 0 \quad (\text{D.6})$$

(b) Eigenfunctions along boundary z, ϕ vanishing at $z = 0$ and $z = L$ and 2π periodic in ϕ

$$\psi_{nm}(z, \phi) = \sin(k_n z) e^{im\phi} \quad k_n \equiv \frac{n\pi}{L} \quad n = 1 \dots \infty \quad m = -\infty \dots \infty$$

(c) Orthogonality:

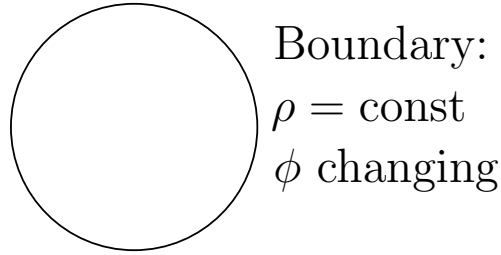
$$\int_0^L dz \int_0^{2\pi} \psi_{nm}(z, \phi) \psi_{n'm'}(z, \phi) = \frac{L}{2} (2\pi) \delta_{nn'} \delta_{mm'}$$

(d) Solution:

$$\Phi = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} [A_{nm} I_m(k_n \rho) + B_{nm} K_m(k_n \rho)] \psi_{nm}(z, \phi) \quad (\text{D.7})$$

Here $I_\nu(x)$ and $K_\nu(x)$ is the modified bessel function of the first and second kinds. Note that $K_{-m}(x) = K_m(x)$ and $I_{-m}(x) = I_m(x)$

D.4 2D cylindrical coordinates



(a) Laplacian:

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right] \Phi = 0 \quad (\text{D.8})$$

(b) Eigenfunctions along boundary ϕ : 2π periodic in ϕ

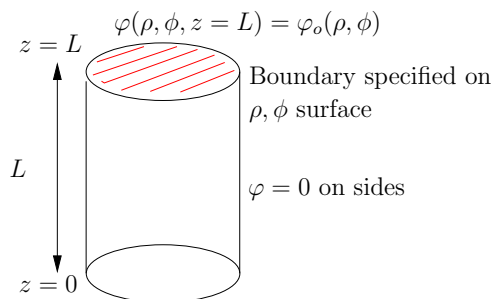
$$\psi_m(\phi) = e^{im\phi} \quad m = -\infty \dots \infty$$

(c) Orthogonality

$$\int_0^{2\pi} \psi_m^*(\phi) \psi_{m'}(\phi) = 2\pi \delta_{mm'} \quad (\text{D.9})$$

(d) Solution

$$\Phi = A_0 + B_0 \ln \rho + \sum_{m=-\infty}^{\infty} \left(A_m \rho^{|m|} + \frac{B_m}{\rho^{-|m|}} \right) \psi_m$$

D.5 Cylindrical Boundary: ρ, ϕ are the boundary

(a) Laplacian:

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right] \Phi = 0 \quad (\text{D.10})$$

(b) Eigenfunctions along boundary ρ, ϕ vanishing at $\rho = R$ and regular at $\rho = 0$, 2π periodic in ϕ :

$$\psi_{mn}(\rho, \phi) = J_m(k_{mn}\rho) e^{im\phi} \quad n = 1 \dots \infty \quad m = -\infty \dots \infty$$

Here:

$$k_{mn} = \frac{x_{mn}}{R} \quad (\text{D.11})$$

where x_{mn} is the n -th zero of the m -th Bessel function, e.g. the zeros of $J_0(x)$ are

$$(x_{01}, x_{02}, x_{03}) = 2.40483, 5.52008, 8.65373 \quad (\text{D.12})$$

These are given by $x_{mn} = \text{BesselZeroJ}[m, n]$ in Mathematica. Note also that $J_{-m}(x) = J_m(x)$

(c) Orthogonality:

$$\int_0^R \rho d\rho \int_0^{2\pi} \psi_{mn}(\rho, \phi) \psi_{m'n'}(\rho, \phi) = \left(\frac{R^2}{2} [J_{m+1}(k_{mn}R)]^2 \right) (2\pi) \delta_{nn'} \delta_{mm'}$$

(d) Solution:

$$\Phi = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} [A_{mn} e^{-k_{mn}z} + B_{nm} e^{k_{mn}z}] \psi_{mn}(\rho, \phi) \quad (\text{D.13})$$

D.6 Continuum Forms and Fourier and Hankel Transforms

In each case we are expanding two directions of the solution in a complete set of eigenfunctions

$$\langle x|F\rangle = \frac{1}{C_n} \sum_n \langle x|n\rangle \langle n|F\rangle, \quad (\text{D.14})$$

and solving the laplace equation to find the dependence on the third direction.

- (a) For the cartesian case when a and b go to infinity. The sum becomes an integral and the sum over n and m becomes a 2D fourier transform

$$\Phi = \int \frac{d^2 \mathbf{k}_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} [A(\mathbf{k}_\perp) e^{-k_\perp z} + B(\mathbf{k}_\perp) e^{k_\perp z}].$$

We are using the fact that any function in the x, y plane (in particular the boundary condition $\Phi_o(x, y)$) can be expressed as a fourier transform pairs

$$F(x, y) \equiv \int \frac{d^2 \mathbf{k}_\perp}{(2\pi)^2} [e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}] F(k_x, k_y), \quad (\text{D.15})$$

$$F(k_x, k_y) \equiv \int d^2 \mathbf{x}_\perp [e^{-i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}] F(x, y). \quad (\text{D.16})$$

- (b) For the cylindrical case when L goes to ∞ , the sum over n becomes an integral yielding

$$\Phi = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} [e^{i\kappa z} e^{im\phi}] [A(\kappa) I_m(|\kappa|\rho) + B(\kappa) K_m(|\kappa|\rho)]$$

We are using the fact that any regular function of z and ϕ (in particular the boundary condition $\Phi_o(z, \phi)$) can be written in terms of its fourier components

$$F(z, \phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} [e^{i\kappa z} e^{im\phi}] F_m(\kappa) \quad (\text{D.17})$$

$$F_m(\kappa) = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz [e^{-i\kappa z} e^{-im\phi}] F(z, \phi) \quad (\text{D.18})$$

- (c) Finally for the second cylindrical case when the radius goes to infinity

$$\Phi = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} k dk [J_m(k\rho) e^{im\phi}] [A(k) e^{-kz} + B(k) e^{kz}] \quad (\text{D.19})$$

We are using the fact that any regular cylindrical function of ρ and ϕ (in particular the boundary condition $\Phi_o(\rho, \phi)$) can be written as *Hankel* transform

$$F(\rho, \phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} k dk [J_m(k\rho) e^{im\phi}] F_m(k) \quad (\text{D.20})$$

$$F_m(k) = \int_0^{2\pi} d\phi \int_0^{\infty} \rho d\rho [J_m(k\rho) e^{-im\phi}] F(\rho, \phi) \quad (\text{D.21})$$