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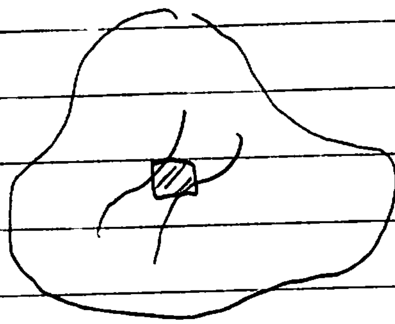
Energy Conservation for Linear Matter:



The energy density

- u_{mech} = mechanical energy/vol (compression, thermal)
- u_{em} = The electric and magnetic energy/vol
- u_{TOT} = total $= u_{mech} + \underbrace{u_{em}}_{\frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{B}\cdot\vec{H}}$

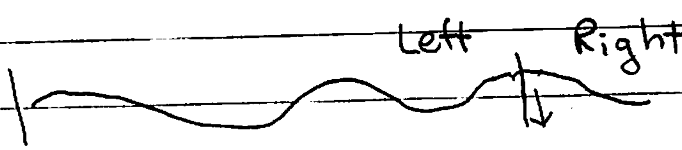
- For the total energy flux Expect:

$$\partial_t u_{TOT} + \partial_i S^i_{TOT} = 0$$

In this way an isolated system will conserve energy.

- The energy flux $\vec{S}_{TOT}$ has mechanical pieces, $\vec{S}_{mech}$, and electromagnetic flux

<u>Ex</u>: Stretched string



$$u_{mech} = \frac{1}{2} T_0 \left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{2} \rho \left(\frac{\partial y}{\partial t}\right)^2$$

$$\vec{S} = \underbrace{T_0 \frac{\partial y}{\partial x}}\ \underbrace{\frac{\partial y}{\partial t}} = \underbrace{\text{Force}} \cdot \underbrace{\text{velocity}}$$

S_{mech} records how energy is mechanically transported from one region to another

We will show:

$$\partial_t (u_{mech} + u_{em}) + \partial_i (S^i_{mech} + S^i_{em}) = 0$$

Where $u_{em} = \frac{1}{2} \vec{E} \cdot \vec{D} + \frac{1}{2} \vec{B} \cdot \vec{H}$

and $\vec{S}_{em} = c(\vec{E} \times \vec{H})$

energy flux in electromagnetism

In integral form:

$$\frac{d(U_{mech} + U_{em})}{dt} = -\int \vec{S}_{em} \cdot d\vec{a} - \underbrace{\cancel{\int \vec{S}_{mech} \cdot d\vec{a}}}_{0}$$

0 for a mechanically isolated system

Then

$$[S] = \frac{\text{energy}}{\text{vol}} \cdot \frac{m}{s} = \frac{\text{energy}}{\text{area } s}$$

Prf - Imagine a charged string

$$\boxed{\partial_t u_{mech} + \partial_i S^i_{mech} = \vec{j} \cdot \vec{E}}$$

Now

$$\vec{j} \cdot \vec{E} = c \left(\nabla \times H - \frac{1}{c} \partial_t \vec{D} \right) \cdot \vec{E}$$

$$= \underbrace{c (\nabla \times H) \cdot E} - E \, \partial_t \vec{D}$$

equivalent

$$= \nabla \cdot (c H \times E) + H \, c \underbrace{\nabla \times E}_{= -\frac{1}{c} \partial_t \vec{B}}$$

$$= -\nabla \cdot S_{em} - (E \partial_t D + H \partial_t B)$$

$$= -\nabla \cdot S_{em} - \partial_t u_{em}$$

And So

$$\boxed{\partial_t (u_{mech}) + \partial_i S^i = -\partial_t u_{em} - \partial_i S^i_{em}}$$

$$u_{em} = \frac{1}{2} E \cdot D + \frac{1}{2} B \cdot H$$

$$\vec{S}_{em} = c \, \vec{E} \times \vec{H}$$

Momentum Conservation

$$\partial_t g^j_{TOT} + \partial_i T^{ij}_{TOT} = 0$$

- g_{TOT} is the total momentum per volume
- T^{ij}_{TOT} is the force in the i-th derecton per area in j-th
- This guarantees that the total momentum is conserved

$$\frac{dP^j_{TOT}}{dt} = \int dV \, \partial_t g^j_{TOT} = \int dV \, [-\partial_i T^{ij}]$$

$$= -\int_{\partial V} T^{ij} n_j \, dS = 0$$

for an isolated system

Will show that

$$\partial_t g^j_{mech} + \partial_i T^{ij}_{mech} = \rho \vec{E} + \frac{\vec{j}}{c} \times \vec{B}$$

$$= -\partial_t (\vec{g}^j_{em}) - \partial_i T^{ij}_{em}$$

same

Where

$$\vec{g}_{em} = \frac{\varepsilon \mu}{c^2} \vec{S}_{em} \qquad T^{ij}_{em} = \overbrace{-\varepsilon E^i E^j + \frac{1}{2} \varepsilon E^2 \delta_{ij}}^{\text{electric stress}}$$

$$+ \underbrace{-\frac{B^i B^j}{\mu} + \frac{1}{2} \frac{B^2}{\mu} \delta_{ij}}_{\text{magnetic stress}}$$

So that the full result:

$$\partial_t(\vec{g}_{mech} + \vec{g}_{em}) + \partial_i(T^{ij}_{mech} + T^{ij}_{em}) = 0$$

Prf

$$\partial_t g^j_{mech} + \frac{\partial T^{ij}_{mech}}{\partial x^i} = f^j_{em}$$

Now write

$$f^j_{em} = \rho E^j + \left(\frac{\vec{j}}{c} \times \vec{B}\right)^j$$

Then use $\nabla \cdot D = \rho$ $\quad \frac{\vec{j}}{c} = \nabla \times \vec{H} - \frac{1}{c}\partial_t \vec{D}$

Find

$$f^j_{em} = \underbrace{(\nabla \cdot D) E^j}_{1} + \underbrace{\left[(\nabla \times \vec{H}) \times \vec{B}\right]^j}_{2} - \underbrace{\frac{1}{c}(\partial_t \vec{D} \times \vec{B})^j}_{3}$$

The rest is labor wich I will not go through (see Jackson)

$$(1) \implies -\partial_i T^{ij}_E \quad \text{with} \quad T^{ij}_E = -\varepsilon E^i E^j + \frac{1}{2}\varepsilon E^2 \delta^{ij}$$

$$(2) \implies -\partial_i T^{ij}_B \quad \text{with} \quad T^{ij}_B = -\frac{B^i B^j}{\mu} + \frac{1}{2}\frac{B^2}{\mu}\delta^{ij}$$

$$(3) \implies -\partial_t \vec{g}_{em} \quad \text{with} \quad \vec{g}_{em} = \vec{D}\times\vec{B} = \frac{\varepsilon\mu}{c^2}\vec{S}$$

(plus a term from (1))

The result is: $\boxed{\rho \vec{E}^j + J/c \times \vec{B} = -\partial_t g^j_{em} - \partial_i T^{ij}_{em}}$

$$\boxed{\begin{aligned} &\partial_t(g^j_{mech} + g^j_{em}) + \partial_i(T^{ij}_{mech} + T^{ij}_{em}) = 0 \\ &\partial_t g^j_{mech} + \partial_i T^{ij}_{mech} = \rho E^j + \left(\frac{\vec{J}}{c}\times\vec{B}\right)^j \end{aligned}} \quad \text{same thing}$$

Or in terms of integralts:

$$\underbrace{\boxed{\frac{d(P^j_{mech} + P^j_{em})}{dt} = -\int da\, n_i T^{ij}_{em}}}_{\text{Total momentum}} \quad \underbrace{-\cancel{\int da\, n_i T^{ij}_{mech}}}_{0 \text{ for a mechanically isolated syste}}$$

$$\vec{P} = \int dV \, \vec{g}$$

Angular Momentum Conservation

$$\partial_t g^k + \partial_\ell T^{\ell k} = 0 \quad \Leftarrow \text{momentum conservation}$$

And take $T^{\ell k} = T^{k\ell}$ symmetric

$$\underbrace{\partial_t ((r\times g)_i)}_{\text{ang momentum}} + \varepsilon_{ijk}\, r^j \partial_\ell T^{\ell k} = 0$$

Now $\varepsilon_{ijk}\, r^j \partial_\ell T^{\ell k} = \varepsilon_{ijk}\left[\partial_\ell (r^j T^{\ell k}) - \underbrace{\overset{\frac{\partial r^j}{\partial r^\ell}}{\delta^j_\ell}\, T^{\ell k}}_{= T^{jk}}\right]$

So since $\varepsilon_{ijk} T^{jk} = 0$, we have

$$\boxed{\partial_t ((\vec{r}\times\vec{g})_i) + \partial_\ell (\varepsilon_{ijk}\, r^j T^{k\ell}) = 0}$$

So the angular momentum of the system is conserved, provided the stress tensor is symmetric

$$\boxed{\vec{L}_{field} = \int_V \vec{r}\times\vec{g}_{em}}$$

$$\frac{d}{dt}\left(\vec{L}_{mech} + \vec{L}_{field}\right)_i = \underbrace{-\int_{\partial V} dS\ \varepsilon_{ijk}\, r^j\, T^{kl}\, n_l}_{\text{net torque exerted on system}}$$