Dimensional Analysis of Maxwell Eqs

Examining the Maxwell Eqs (In heay side- lorentz units)

\[ \nabla \cdot E = \rho \]
\[ \nabla \times B = \frac{i}{c} + \frac{1}{c} \frac{\partial E}{\partial t} \]
\[ \nabla \cdot B = 0 \]
\[ - \nabla \times E = \frac{\partial B}{\partial t} \]

\[ \mathbf{J} = \frac{\text{charge}}{\text{volume}} \times \frac{\text{Velocity}}{c} \]

*Note every time derivative comes with \( \frac{1}{c} \), i.e. \( \frac{\partial}{\partial t} \)

Every velocity is measured in units of \( c \)

We see that if the system has characteristic length \( L \) and characteristic time scale \( T \), then the solutions will change rather dramatically from when

\[ \frac{L}{c} \ll T \quad \text{vs.} \quad T \ll \frac{L}{c} \]

This regime is radiation dominated. The system evolves significantly over the time it takes for the fields to propagate (at \( c \)) across the system.

---

this is the regime of electrostatics, magnetostatics, and quasi-statics. The fields very rapidly adjust (with speed \( c \)) to changes of charges.
Consider the quasi-static regime:

\[ \frac{L}{c^2} \ll \frac{L}{cT} \sim 10^{-8} \text{ say.} \]

Then the different terms in the Maxwell equations have very different magnitudes.

For instance:

\[ \nabla \cdot \mathbf{E} \sim \frac{\mathbf{E}}{L} \]

while

\[ \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \sim \frac{\mathbf{E}}{cT} \sim \frac{L}{cT} \left( \frac{\mathbf{E}}{L} \right) \ll \nabla \cdot \mathbf{E} \]

Thus we should set up a series solution in powers of \( \frac{L}{c} \).

Most undergraduate courses stay entirely in this approximation scheme (without telling you).
Set up a series in $1/c$

\[ E = E^{(0)} + E^{(1)} + E^{(2)} + \ldots \]

\[ B = B^{(0)} + B^{(1)} + B^{(2)} \]

where $E^{(1)} \sim 10^{-8} E^{(0)}$ and $E^{(2)} \sim 10^{-16} E^{(0)}$ etc.

Then substituting this series into the Maxwell equations we find to zeroth order:

\[ \nabla \cdot E^{(0)} = \rho \quad \text{This is electrostatics, } B^{(0)} = 0 \]
\[ \nabla \times B^{(0)} = 0 \quad \text{and} \]
\[ \nabla \cdot B^{(0)} = 0 \]
\[ -\nabla \times E^{(0)} = 0 \quad \nabla \cdot E = \rho \]
\[ \nabla \times E = 0 \]

At first order:

\[ \nabla \cdot E^{(1)} = 0 \quad \text{determined from electrostatics, } E^{(1)} = 0 \]
\[ \nabla \times B^{(1)} = \frac{j}{c} + \frac{1}{c} \frac{\partial E^{(0)}}{\partial t} \quad \text{This is magneto-}
\]
\[ \text{statics. } E^{(1)} = 0 \]
\[ \nabla \cdot B^{(1)} = 0 \quad \text{and } \nabla \times B \text{ solves:} \]
\[ \nabla \times B = \frac{j}{c} + \frac{1}{c} \frac{\partial E^{(0)}}{\partial t} \]
\[ \nabla \cdot B = 0 \]
\[ \nabla \times E^{(1)} = \frac{\partial B^{(0)}}{\partial t} \]
One can continue this way and find corrections to electrostatics and magneto-statics. We will do this later in the course.