Dimensional Analysis of Maxwell Eqs

Examining the Maxwell Eqs (In heavyside-lorentz units)

$$\nabla \cdot E = \rho$$

$$\nabla \times B = \frac{j}{c} + \frac{1}{c}\frac{\partial E}{\partial t}$$

$$\nabla \cdot B = 0$$

$$-\nabla \times E = \frac{1}{c}\frac{\partial B}{\partial t}$$

- Note every time derivatice comes with $1/c$, i.e $\frac{1}{c}\frac{\partial}{\partial t}$
- Every velocity is measured in units of c

$$\frac{j}{c} = \frac{\text{Charge}}{\text{Volume}} \times \frac{\text{velocity}}{c}$$

We see that if the system has characteristic length L, and characteristic time scale T, then the solutions will change rather dramatically from when

$$\underbrace{\frac{L}{c} \ll T}_{} \qquad \text{vs.} \qquad \underbrace{T \ll \frac{L}{c}}_{}$$

this is the regime of electrostatics, magnetostatics, and quasi-statics. The fields very rapidly adjust (with speed c) to changes of charges.

This regime is radiation dominated. The system evolves significantly over the time it takes for the fields to propagate (at c) across the system.

Consider the quasi-static regime:

$$\frac{L}{T} \ll c \,, \qquad \frac{L}{cT} \sim 10^{-8} \text{ say.}$$

Then the different terms in the maxwell equations have very different magnitudes.

For instance:

$$\nabla \cdot E \sim \frac{E}{L}$$

while

$$\frac{1}{c}\frac{\partial E}{\partial t} \sim \frac{E}{cT} \sim \frac{L}{cT}\left(\frac{E}{L}\right) \ll \nabla \cdot E$$

Thus we should set up a series solution in powers of $\frac{1}{c}$.

Most undergraduate courses stay entirely in this approximation scheme (without telling you).

Set up a series in $1/c$

$$E = E^{(0)} + E^{(1)} + E^{(2)} + \ldots$$

$$B = B^{(0)} + B^{(1)} + B^{(2)}$$

Where $E^{(1)} \sim 10^{-8} E^{(0)}$ and $E^{(2)} \sim 10^{-16} E^{(0)}$ etc.

Then substituting this series into the maxwell equations we find to zeroth order:

$$\nabla \cdot E^{(0)} = \rho$$
$$\nabla \times B^{(0)} = 0$$
$$\nabla \cdot B^{(0)} = 0$$
$$-\nabla \times E^{(0)} = 0$$

This is electrostatics, $B^{(0)} = 0$ and

$$\nabla \cdot E = \rho$$
$$\nabla \times E = 0$$

At first order:

$$\nabla \cdot E^{(1)} = 0$$

determined from electro statics

$$\nabla \times B^{(1)} = \frac{j}{c} + \frac{1}{c}\frac{\partial E^{(0)}}{\partial t}$$

$$\nabla \cdot B^{(1)} = 0$$

$$\nabla \times E^{(1)} = \frac{1}{c}\cancel{\frac{\partial B^{(0)}}{\partial t}} = 0$$

This is magneto-statics. $E^{(1)} = 0$ and $\vec{B}$ solves:

$$\nabla \times B = \vec{j}/c + \frac{1}{c}\frac{\partial E^{(0)}}{\partial t}$$

$$\nabla \cdot B = 0$$

One can continue this way and find corrections to electrostatics and magneto-statics. We will do this later in the course.