1 Radiation from a relativistic electron

Consider a relativistic electron (of charge e) traveling with an initial speed of v_o along the z-axis. At time $t = 0$ it slows down to a stop over a time τ while moving along the z-axis

$$v(t) = v_o \left(1 - \frac{t}{\tau}\right), \qquad 0 \le t \le \tau\,. \tag{1}$$

Recall that the electric field in the far field radiated from a point charge following a trajectory with position $\boldsymbol{x}(t)$, and velocity $\boldsymbol{v}(T) = \boldsymbol{x}'(t)$ is

$$\boldsymbol{E}_{\rm rad}(t, \boldsymbol{r}) = \frac{e}{4\pi c^2} \left[\frac{\boldsymbol{n} \times (\boldsymbol{n} - \boldsymbol{\beta}) \times \boldsymbol{a}}{R\,(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3} \right]_{\rm ret}, \tag{2}$$

where all quantities in square brackets are evaluated at the *retarded time*, $T(t, \boldsymbol{r})$ (which you will define below). The other symbols are defined as $\boldsymbol{n} \equiv (\boldsymbol{r} - \boldsymbol{x}(T))/|\boldsymbol{r} - \boldsymbol{x}(T)|$, $R \equiv |\boldsymbol{r} - \boldsymbol{x}(T)|$, and $\boldsymbol{\beta} = \boldsymbol{v}/c$.

(a) (3 points) Define the retarded time and compute the derivatives $\partial T/\partial t$ and $\partial T/\partial \boldsymbol{r}^i$

(b) (3 points) The radiation field $\boldsymbol{E}_{\rm rad}$ is derived from the *Lienard-Wiechert* potentials

$$\varphi(t, \boldsymbol{r}) = \frac{e}{4\pi} \left[\frac{1}{R(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})} \right]_{\rm ret}, \tag{3}$$

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{e}{4\pi c} \left[\frac{\boldsymbol{v}}{R(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})} \right]_{\rm ret}. \tag{4}$$

Using far field approximations, show that the Lorenz gauge condition is satisfied by these potentials.

(c) (6 points) For the decelerating electron described above, compute:

(i) the energy radiated per solid angle per *retarded time*.

(ii) the energy radiated per solid angle *per time*.

Describe in what physical situations you would be interested in (i) and (ii) respectively. Use no more than two sentences to describe each case.

(d) (4 points) Now consider a relativistic electron with initial energy of 1 GeV.

Examining your results of part (c), you should find that at $t = 0$ the radiation is initially emitted (predominantly) at a characteristic angle. Give an order of magnitude estimate for this angle. Explain your estimate by pointing to specific terms in the formula from part (c).

(e) (4 points) Determine the total energy per solid angle emitted as the electron decelerates to a stop.

Solution

(a) The retarded time is the time that light was emitted at the source such that it arrives at space-time observation point $(t, \boldsymbol{r})$. It satisfies the implicit equation

$$t - T = |\boldsymbol{r} - \boldsymbol{x}(T)|/c\,. \tag{5}$$

Differentiating

$$1 - \frac{\partial T}{\partial t} = -\frac{(\boldsymbol{r} - \boldsymbol{x}(T))^\ell}{|\boldsymbol{r} - \boldsymbol{x}(T)|} v_\ell(T)/c \frac{\partial T}{\partial t}\,, \tag{6}$$

$$1 - \frac{\partial T}{\partial t} = -\,\boldsymbol{n} \cdot \beta(T) \frac{\partial T}{\partial t}\,. \tag{7}$$

Thus

$$\frac{\partial T}{\partial t} = \frac{1}{1 - \boldsymbol{n} \cdot \boldsymbol{\beta}(T)}\,. \tag{8}$$

Similarly,

$$-\frac{\partial T}{\partial r^k} = \frac{(\boldsymbol{r} - \boldsymbol{x}(T))^\ell}{|\boldsymbol{r} - \boldsymbol{x}(T)|} \left(\delta_{\ell k} - \frac{v_o(T)_\ell}{c} \frac{\partial T}{\partial r^k} \right)\,. \tag{9}$$

Thus

$$\frac{\partial T}{\partial r^k} = \frac{-n_k}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta}(T))}\,. \tag{10}$$

(b) The Lorenz gauge condition reads

$$\frac{1}{c} \partial_t \varphi + \partial_i A^i = 0\,. \tag{11}$$

In the far field we neglect differentiating $1/R$ and $\boldsymbol{n}$ which lead to subleading terms in $1/R$. Then in the far field we differentiate

$$\frac{1}{c} \partial_t \varphi = \frac{e}{4\pi R c^2} \frac{\boldsymbol{n} \cdot \boldsymbol{a}}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^2} \frac{\partial T}{\partial t}\,, \tag{12}$$

$$= \frac{e}{4\pi R c^2} \frac{\boldsymbol{n} \cdot \boldsymbol{a}}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3}\,. \tag{13}$$

Similarly,

$$\partial_i \boldsymbol{A}^i = \frac{e}{4\pi R c^2} \left[\frac{a^i}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})} \frac{\partial T}{\partial r^i} + \frac{\beta^i}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^2} (\boldsymbol{n} \cdot \boldsymbol{a}) \frac{\partial T}{\partial r^i} \right]\,, \tag{14}$$

$$= \frac{e}{4\pi R c^2} \left[\frac{-\boldsymbol{n} \cdot \boldsymbol{a}}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^2} + \frac{-\boldsymbol{n} \cdot \beta}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3} (\boldsymbol{n} \cdot \boldsymbol{a}) \right]\,, \tag{15}$$

$$= \frac{e}{4\pi R c^2} \left[\frac{-\boldsymbol{n} \cdot \boldsymbol{a}}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3} \right]\,. \tag{16}$$

So we verify that

$$\frac{1}{c} \partial_t \varphi + \partial_i \boldsymbol{A}^i = 0\,. \tag{17}$$

(c) In this case $\boldsymbol{\beta} \times \boldsymbol{a} = 0$, $|\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{a}| = a\sin(\theta)$, and thus the magnitude of $\boldsymbol{E}$ is

$$E = \frac{e}{4\pi Rc^2} \frac{a\sin\theta}{(1-\beta(T)\cos\theta)^3} \tag{18}$$

So the energy per time per solid angle

$$\frac{dW}{dtd\Omega} = \lim_{r\to\infty} c|r\boldsymbol{E}|^2 \tag{19}$$

$$= \frac{e}{(4\pi)^2 c^3} \frac{a^2\sin^2\theta}{(1-\beta(T)\cos\theta)^6} \tag{20}$$

where $a = v_o/\tau$, and $\beta(T) = \beta_o(1-T/\tau)$. The energy *per retarded time* per solid angle is

$$\frac{dW}{dTd\Omega} = \frac{dW}{dtd\Omega}\frac{dt}{dT} \tag{21}$$

$$= \frac{e^2}{(4\pi)^2 c^3} \frac{a^2\sin^2\theta}{(1-\beta(T)\cos\theta)^5} \tag{22}$$

The energy per time is useful if you want to know whether a remote detector will burn up. The energy per retarded time is useful if you want to calculate how much energy is lost to radiation over a given element of a particles trajectory, $d\boldsymbol{x} = \boldsymbol{v}(T)dT$.

(d) We see that the denominator function, $1-\beta_o\cos\theta$, is approaching zero at small angle since $\beta_o \simeq 1$. Expanding $\beta_o \simeq 1 - \frac{1}{2\gamma^2}$ and $\cos\theta \simeq 1 - \frac{\theta^2}{2}$,

$$\frac{1}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}} \simeq \frac{1}{\frac{1}{2\gamma_o^2}+\frac{\theta^2}{2}} = \frac{2\gamma_o^2}{1+(\gamma_o\theta)^2}\,. \tag{23}$$

So the characteristic angle is $\theta \sim 1/\gamma_o$. For a 1 GeV electron, $\gamma \simeq E/m_e c^2 \sim 2000$. So $\theta \sim 1/2000$.

(e) The total energy is

$$\frac{dW}{d\Omega} = \int_0^\tau dT \frac{dW}{dTd\Omega}\,. \tag{24}$$

So with the result of Eq. 21 we have

$$\frac{dW}{d\Omega} = \frac{e^2}{(4\pi)^2c^3}(a^2\sin^2\theta)\int_0^\tau dT \frac{1}{(1-\beta_o(1-\frac{T}{\tau})\cos\theta)^5}\,, \tag{25}$$

$$= \frac{e^2}{(4\pi)^2c^3}\frac{\tau(a^2\sin^2\theta)}{4\beta_o\cos\theta}\left[\frac{-1}{(1-\beta_o(1-\frac{T}{\tau})\cos\theta)^4}\right]_0^\tau, \tag{26}$$

$$= \frac{e^2}{(4\pi)^2c^3}\frac{\tau(a^2\sin^2\theta)}{4\beta_o\cos\theta}\left[\frac{1}{(1-\beta_o\cos\theta)^4}-1\right]. \tag{27}$$

In the ultra relativistic limit we have

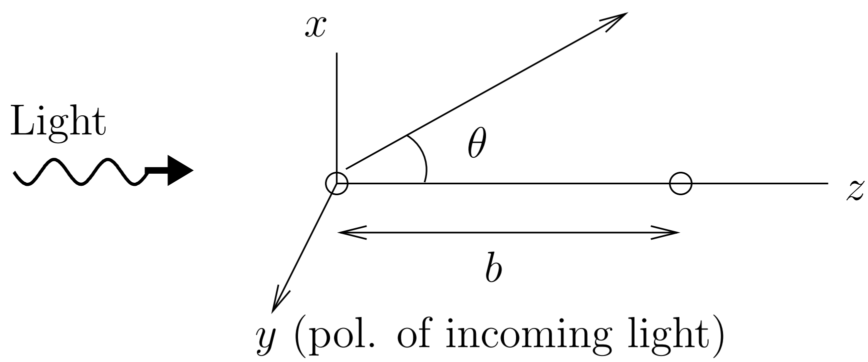
$$\frac{1}{1-\beta_o\cos\theta} \simeq \frac{1}{\frac{1}{2\gamma_o^2}+\frac{\theta^2}{2}} = \frac{2\gamma_o^2}{1+(\gamma_o\theta)^2}\,, \tag{28}$$

and thus

$$\frac{dW}{d\Omega} \simeq \frac{e^2a^2\tau}{(4\pi)^2c^3}\,4\gamma_o^2\left[\frac{(\gamma_o\theta)^2}{(1+(\gamma_o\theta)^2)^4}\right]\,. \tag{29}$$

2 Scattering at different scales

Consider the scattering of an electromagnetic plane wave of wavenumber k and frequency ω propagating in the z direction. The incident light is linearly polarized in the y direction, $\boldsymbol{E}(t, \boldsymbol{r}) = \hat{\boldsymbol{y}} E_0 e^{ikz - i\omega t}$. The light is scattered by two small dielectric spheres of radius a separated by a distance b with $b \gg a$. The first sphere is centered at the origin, while the second sphere is located on the z axis with $z = b$. The two spheres have dielectric constant $\epsilon = 1 + \chi$ with $\chi \ll 1$.



(a) (5 points) Consider the scattering of long wavelength light $kb \ll 1$. Determine the total cross section of the two spheres to leading order in kb.

(i) How does the cross section of the two spheres compare to the cross section of a single sphere?

(b) (5 points) Remaining in the long wavelength limit $kb \ll 1$, determine the electric field as a function of time at a specific point along the x axis, $\boldsymbol{r} = (x, y, z) = (2b, 0, 0)$. Hint: is this point in the near or far field?

(c) (5 points) Now consider the scattering of shorter wavelength light with $kb \sim 1$ but still $ka \ll 1$. Determine the differential cross section $d\sigma/d\Omega$ of the two spheres for light scattered at an angle θ in the z, x plane (see diagram above).

- Sketch the differential cross section $d\sigma/d\Omega$ for scattering at $\theta = \pi/2$ (along the x axis) as a function of k for $kb = 0 \ldots 8\pi$.

(d) (5 points) Now instead of a plane wave of light, consider the scattering of a wave packet with mean wavenumber $\bar{k}$ and bandwidth Δk, with $\Delta k/k \simeq 1/10$. The differential cross section is the energy scattered per solid angle divided by the total energy in the wave packet.

- Qualitatively sketch the differential cross section $d\sigma/d\Omega$ for scattering at $\theta = \pi/2$ as a function of $\bar{k}$, and contrast this sketch with the $\Delta k = 0$ limit drawn in the second part of (c). At large k how does the cross section for the two spheres compare to the cross section for one sphere?

Solution

(a) To leading order in kb the external field induces an identical dipole moment in each sphere of magnitude $\chi V E_0$. The two dipoles then radiate electromagnetic radiation via dipole radiation The total electric dipole moment of the *two* spheres is

$$\boldsymbol{p} = 2\chi V E_0 e^{-i\omega t}\,\hat{\text{y}} \tag{30}$$

The radiated power for dipole radiation

$$P = \frac{\omega^4}{4\pi c^3}\frac{|\boldsymbol{p}|^2}{3} \tag{31}$$

So the cross section is

$$\sigma = \frac{P}{\frac{c}{2}|E_0|^2} = \frac{\omega^4}{4\pi c^2}\frac{2}{3}|p_0|^2 \tag{32}$$

Collecting all factors

$$\sigma = \frac{P}{\frac{c}{2}|E_0|^2} = \frac{\omega^4}{4\pi c^4}\frac{8}{3}\chi^2 V^2 \tag{33}$$

(i) The total dipole moment is twice as large as a single sphere. The cross goes as the square of the dipole moment and is therefore four times as large

(b) This is in the near field. The electric field is just the electric field of two dipoles, one situated at the origin and one situated at $z = b$. The field from an electric dipole is

$$\boldsymbol{E} = \frac{3(\boldsymbol{p}\cdot\boldsymbol{n})\boldsymbol{n} - \boldsymbol{p}}{4\pi r^3}\,. \tag{34}$$

where $\boldsymbol{n}$ is the vector from the dipole origin to the observation point, and $\boldsymbol{p}$ is the dipole moment. In the current setup, $\boldsymbol{p}$ points in the y direction and $\boldsymbol{n}$ lies in the x, z plane for both dipoles. Thus the sum of the fields from the two dipoles is

$$\boldsymbol{E} = \frac{-\boldsymbol{p}}{4\pi r_1^3} + \frac{-\boldsymbol{p}}{4\pi r_2^3} \tag{35}$$

where r_1 and r_2 are the distances to the two induced dipole moments.

$$r_1 = 2b \tag{36}$$

$$r_2 = \sqrt{(2b)^2 + b^2} = \sqrt{5}b \tag{37}$$

This leads to

$$\boldsymbol{E}(t) = -\hat{\boldsymbol{y}}\frac{\chi V}{4\pi b^3}E_0 e^{-i\omega t}\left[\frac{1}{8} + \frac{1}{5\sqrt{5}}\right] \tag{38}$$

(c) In this case the two dipoles are out of phase

$$\boldsymbol{p}_1 = \hat{\boldsymbol{y}}\chi V E_0 e^{-i\omega t} \tag{39}$$

$$\boldsymbol{p}_2 = \hat{\boldsymbol{y}}\chi V E_0 e^{-i\omega t + ikb} \tag{40}$$

The radiation is

$$\boldsymbol{A}_{\rm rad} = \frac{e^{-i\omega(t-r/c)}}{4\pi r c} \int d^3\boldsymbol{r}_o \boldsymbol{J}(\boldsymbol{r}_o) e^{-ik\boldsymbol{n}\cdot\boldsymbol{r}_o} \tag{41}$$

Thus examining this integral we see that there is an overall phase difference between the two dipoles

$$-k\boldsymbol{n}\cdot\boldsymbol{r}_o = -kb\cos\theta \tag{42}$$

So

$$\boldsymbol{E}_1 = \frac{-\omega^2 e^{-i\omega(t-r/c)}}{4\pi r c^2}\, \boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{p}_1 \tag{43}$$

$$\boldsymbol{E}_2 = \frac{-\omega^2 e^{-i\omega(t-r/c)}}{4\pi r c^2}\, \boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{p}_2 e^{-ikb\cos\theta} \tag{44}$$

For $\boldsymbol{n}$ in the x, z plane and $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ oriented in the y direction we have $\boldsymbol{n}\times\boldsymbol{n}\times\hat{\boldsymbol{y}} = -\hat{\boldsymbol{y}}$

$$\frac{dP}{d\Omega} = \frac{c}{2}|\boldsymbol{E}_1 + \boldsymbol{E}_2|^2 \tag{45}$$

$$= \frac{ck^4}{32\pi^2}(\chi V)^2 E_0^2 \left|1 + e^{ikb(1-\cos\theta)}\right|^2 \tag{46}$$

So the cross section is

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2}(\chi V)^2 \left|1 + e^{ikb(1-\cos\theta)}\right|^2 . \tag{47}$$

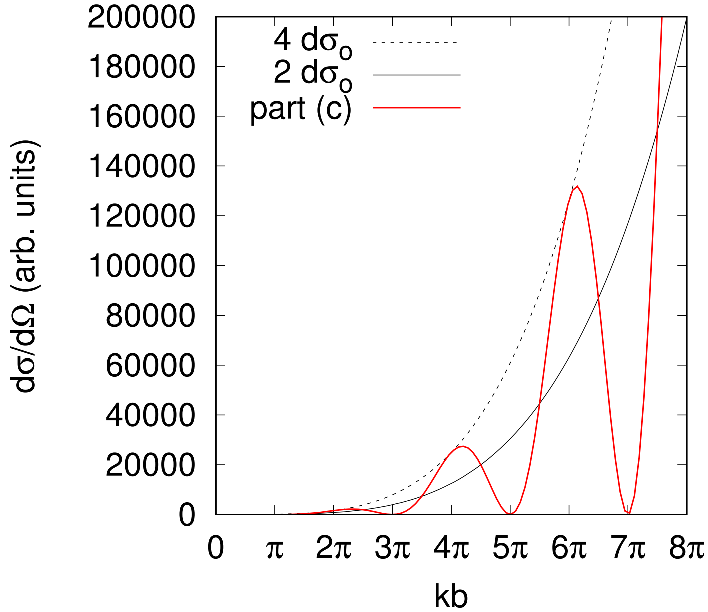
To make a graph we first note that

$$\left|1 + e^{ikb(1-\cos\theta)}\right|^2 = |2\cos(kb\sin^2(\theta/2))|^2 . \tag{48}$$

Thus at $\theta = \pi/2$ we are plotting

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta=\pi/2} \propto (kb)^4 \cos^2(kb/2)\,, \tag{49}$$

We graph this function below

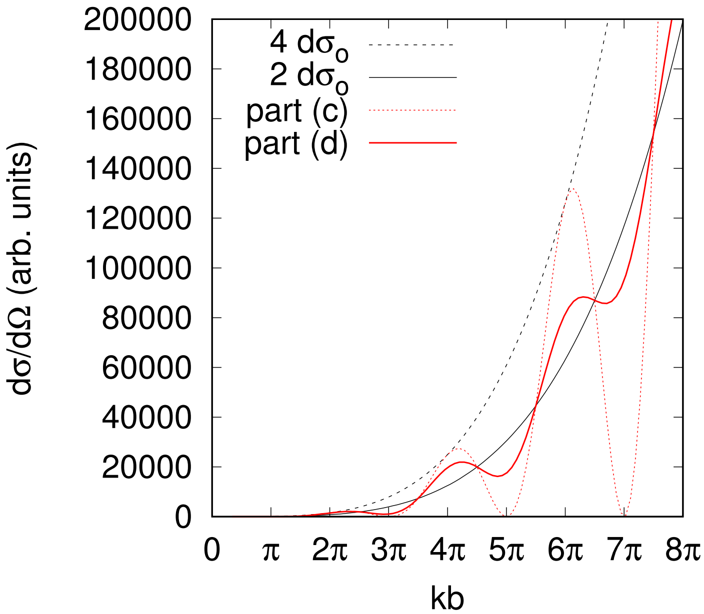


(d) If the wave packet has a finite band width Δk, it has a spatial size of order $\Delta x \sim \frac{1}{\Delta k}$. When this size comes comparable to the spacing b, i.e. $\Delta x \sim b$ or $\Delta k\, b \sim 1$, the interference between the scattering centers will not be complete. Indeed, when the mean Fourier compment of the wave packet $\bar{k}b$ is at an interference maximum, most of the Fourier components in the packet, $k \sim (\bar{k}{\pm}\Delta k)\, b$, will not be at an interference maximum if $\Delta k\, b \sim 1$. In the limit $\Delta x \ll b$ (or $\Delta k b \gg 1$) the wave packet will scatter off the first sphere and then scatter of the second sphere. The cross section for scattering off of the two spheres is thus twice the cross section for scattering off one of the spheres in this limit.

Since $\Delta k/\bar{k} \sim 10$, when

$$\bar{k}\, b \sim 10\,, \tag{50}$$

the interference betweeen the two scattering centers will begin to wash out. A schematic sketch of the cross section in this case is shown below. In the coherent limit (part (c)) the cross section varies between zero and four times the cross section for the scattering off a single sphere correspoinding to destructive and constructive interference respectively. If there is a finite coherence length Δx then the cross section transitions from the coherent limit to the incoherent limit. In the incoherent limit the cross section is twice the cross section of a single sphere.


4 dσo
2 dσo
part (c)
part (d)
dσ/dΩ (arb. units)
200000
180000
160000
140000
120000
100000
80000
60000
40000
20000
0
0
π
2π
3π
4π
5π
6π
7π
8π
kb

3 Fields of a non-relativistic particle

A charge particle of charge q moves non-relativistically with trajectory $\boldsymbol{R}(t)$:

(a) (6 pnts) Show that two of the four Maxwell equations are satisfied by expressing the fields $\boldsymbol{E}, \boldsymbol{B}$ in terms of the scalar and vector potentials, $A^\mu = (\varphi, \boldsymbol{A})$. Use the remaining Maxwell equations to derive the equations for the scalar and vector potentials in the Lorentz gauge.

(b) (8 pnts) Recall that the Green function of the wave equation is[1]

$$G(t - t_o, \boldsymbol{r} - \boldsymbol{r}_o) = \frac{\theta(t - t_o)}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|}\delta(t - t_o - \frac{|\boldsymbol{r} - \boldsymbol{r}_o|}{c}) \,. \tag{52}$$

Use this Green function to derive the potential, φ and $\boldsymbol{A}$, that are appropriate in the far field and the non-relativistic limit. Explicitly explain how the non-relativistic and far-field approximations are used at various points in the derivation to arrive at the final result.

(c) (4 pnts) If the particle is speeding up along the z axis

$$\boldsymbol{R}(t) = \left(v_o t + \frac{1}{2}at^2\right)\hat{\boldsymbol{z}} \,,$$

determine the electric field in the far field as measured on the x-axis. What is the polarization of the radiated field when measured on this axis?

(d) (2 pnts) Assuming the motion as in part (c), determine the power radiated per solid angle in the $\hat{\boldsymbol{x}}$ direction.

[1] The Green function satisfies

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right) G(t, \boldsymbol{r}) = \delta(t)\delta^3(\boldsymbol{r}) \tag{51}$$

Solution

(a) The source free Maxwell are satisfied beacuase partial derivatives commute:

$$\partial_i B^i = \partial_i \epsilon^{ijk}\partial_j A_k = 0 \tag{53}$$

$$-\frac{1}{c}\partial_t B^i - (\nabla \times E)^i = -\partial_t \epsilon^{ijk}\partial_j A_k - \epsilon^{ijk}\partial_j \left(-\frac{1}{c}\partial_t A_k - \partial_k \varphi\right) \tag{54}$$

$$= 0 \tag{55}$$

The first sourced maxwell equations

$$-\nabla \cdot \boldsymbol{E} = \rho\,, \tag{56}$$

becomes with $\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\varphi$

$$-\Box\varphi - \frac{1}{c}\partial_t(\frac{1}{c}\partial_t\varphi + \nabla \cdot \boldsymbol{A}) = \rho\,. \tag{57}$$

Then, writing the second sourced maxwell equation

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t \boldsymbol{E} \tag{58}$$

in terms of $\boldsymbol{A}$ and ϕ, using

$$\nabla \times \nabla \times \boldsymbol{A} = -\nabla^2 \boldsymbol{A} + \nabla(\nabla \cdot \boldsymbol{A})\,, \tag{59}$$

yields

$$-\Box\boldsymbol{A} + \frac{1}{c}\nabla(\frac{1}{c}\partial_t\varphi + \nabla \cdot \boldsymbol{A}) = \frac{\boldsymbol{j}}{c}\,. \tag{60}$$

In the Lorentz gauge,

$$\frac{1}{c}\partial_t\varphi + \nabla \cdot \boldsymbol{A} = 0\,, \tag{61}$$

we find two wave equations

$$-\Box\varphi = \rho\,, \tag{62}$$

$$-\Box\boldsymbol{A} = \frac{\boldsymbol{j}}{c}\,. \tag{63}$$

(b) Using the green function of the wave equation

$$\varphi(t, \boldsymbol{r}) = \int dt_o d^3\boldsymbol{r}_o \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|}\delta(t - t_o - \frac{|\boldsymbol{r} - \boldsymbol{r}_o|}{c}) e\delta^3(\boldsymbol{r}_o - \boldsymbol{R}_o(t_o)) \tag{64}$$

$$\boldsymbol{A}(t, \boldsymbol{r}) = \int dt_o d^3\boldsymbol{r}_o \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|}\delta(t - t_o - \frac{|\boldsymbol{r} - \boldsymbol{r}_o|}{c}) e\boldsymbol{v}(t_o)\delta^3(\boldsymbol{r}_o - \boldsymbol{R}_o(t)) \tag{65}$$

Integrating over $\boldsymbol{r}_o$

$$\varphi(t,\boldsymbol{r}) = \int dt_o \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{R}(t_o)|}\delta(t-t_o-\frac{|\boldsymbol{r}-\boldsymbol{R}(t_o)|}{c})\, e \tag{66}$$

$$\boldsymbol{A}(t,\boldsymbol{r}) = \int dt_o d^3\boldsymbol{r}_o \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{R}(t_o)|}\delta(t-t_o-\frac{|\boldsymbol{r}-\boldsymbol{R}(t_o)|}{c})\, e\boldsymbol{v}(t_o) \tag{67}$$

In the far field we approximate

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{R}(t_o)|} \simeq \frac{1}{4\pi r} \tag{68}$$

Integrating over t_o involves

$$\delta(t-t_o-\frac{|\boldsymbol{r}-\boldsymbol{R}(t_o)|}{c}) = \frac{1}{1-\boldsymbol{n}\cdot\beta(T)}\delta(t_o-T)\,. \tag{69}$$

where T (the retarded time) satisfies

$$T = t - \frac{|\boldsymbol{r}-\boldsymbol{R}(T)|}{c} \simeq t - r/c - \frac{\boldsymbol{n}\cdot\boldsymbol{R}(T)}{c} \tag{70}$$

The last approximation is a far field approximation. In a non relativistic limit

$$T \approx t - \frac{r}{c} \tag{71}$$

and

$$\delta(t-t_o-\frac{|\boldsymbol{r}-\boldsymbol{R}(t_o)|}{c}) \approx (1+\boldsymbol{n}\cdot\beta(t-r/c))\delta(t_o-(t-r/c))\,. \tag{72}$$

So to linear order in v/c, we have

$$\boldsymbol{A}(t,\boldsymbol{r}) \simeq \frac{1}{4\pi r} e \frac{\boldsymbol{v}(t-r/c)}{c}\,. \tag{73}$$

For the scalar potential φ, we integrate over t_o and expand to first order in v/c:

$$\varphi(t,\boldsymbol{r}) = \frac{1}{4\pi r}\frac{e}{1-\boldsymbol{n}\cdot\boldsymbol{v}(T)/c} \simeq \frac{e}{4\pi r}(1+\boldsymbol{n}\cdot\boldsymbol{v}(t-r/c)/c)\,. \tag{74}$$

(c) Computing the electric field we have to leading order in $1/r$

$$\boldsymbol{E} = -\frac{1}{c}\partial_t\boldsymbol{A} - \nabla\varphi\,, \tag{75}$$

$$\approx \frac{e}{4\pi r c^2}(-\boldsymbol{a}+\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{a}))\,, \tag{76}$$

where we used

$$\nabla r = \boldsymbol{n}\,, \tag{77}$$

when differentiating the potentials of part (b).

Then for a particle speeding up in the z-direction, $\boldsymbol{n} = \hat{\boldsymbol{x}}$ and we see that $\boldsymbol{E}$ is polarized in the negative z direction. The electric field on the x-axis is

$$\boldsymbol{E} = -\frac{1}{4\pi r c^2} a\,\hat{\boldsymbol{z}} \tag{78}$$

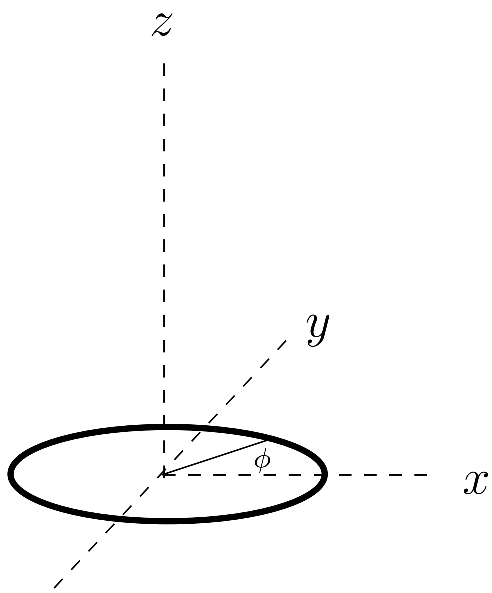
(d) The power radiated on the $x-$axis is

$$\frac{dP}{d\Omega} = c|rE|^2 = \frac{e^2}{16\pi^2 c^3}\, a^2\,. \tag{79}$$

4 EM Comps Problem, Fall 2015, JV

A current is driven through a ring of radius R in the xy plane (see below). Using a complex notation, the current has a harmonic time dependence, $\boldsymbol{J}(t, \boldsymbol{r}) = e^{-i\omega t}\boldsymbol{J}(\boldsymbol{r})$, and the spatial dependence is

$$\boldsymbol{J}(\boldsymbol{r}) = I_0 \sin(\phi)\, \delta(\rho - R)\delta(z)\hat{\boldsymbol{\phi}}\,. \tag{80}$$



1. (4 pnts) Sketch the current flow at time $t = 0$ and $t = \pi/\omega$, and determine the charge density $\rho(t, \boldsymbol{r})$. Show that it corresponds to an oscillating electric dipole, and determine the electric dipole moment.

2. In the long wavelength limit, and in the radiation zone, determine each of the following quantities in the xz plane[2]:

   (a) (6 pnts) The vector potential $\boldsymbol{A}(t, \boldsymbol{r})$ in the Lorentz gauge.

   (b) (4 pnts) The $\boldsymbol{B}(t, \boldsymbol{r})$ field.

   (c) (4 pnts) The (time averaged) angular distribution of the radiated power, $dP/d\Omega$.

3. (2 pnts) What is the polarization of the radiated electric field when viewed along the z axis ?

[2]Specifically compute the fields and power at the spacetime point $\boldsymbol{r} = (x, y, z) = (r\sin\theta, 0, r\cos\theta)$.

Solution

We use Heavyside-Lorentz units.

1. Using current conservation, $\partial_t \rho + \nabla \cdot \boldsymbol{J} = 0$ and a harmonic time dependence, $\rho(t, \boldsymbol{r}) = e^{-i\omega t}\rho(\boldsymbol{r})$,

$$-i\omega\rho(\boldsymbol{r}) = -\nabla \cdot \boldsymbol{J}(\boldsymbol{r}) = -\frac{1}{R}\frac{\partial}{\partial\phi}J^{\phi}\,. \tag{81}$$

Thus

$$\rho(\boldsymbol{r}) = -\frac{I_o \cos\phi}{-i\omega R}\delta(z)\delta(\rho - R) \tag{82}$$

Note, the charge distribution gives rise to a net dipole moment

$$\boldsymbol{p} = \int d^3\boldsymbol{r}\,\rho(\boldsymbol{r})\boldsymbol{r} = \frac{I_o R}{-i\omega}(-\pi\hat{\boldsymbol{x}}) \tag{83}$$

pointed along the negative $\hat{\boldsymbol{x}}$ direction. If this is recognized then the remainder of this problem is just quoting the results of the electric dipole radiation.

2. a) In the dipole approximation we have

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{e^{-i\omega t + ikr}}{4\pi r}\int d^3\boldsymbol{r}'\boldsymbol{J}(\boldsymbol{r}')/c = \frac{e^{-i\omega t + iikr}}{4\pi r}\int \rho d\rho d\phi dz\,\hat{\boldsymbol{\phi}}\,(I_0/c)\sin\phi\,\delta(\rho - R)\delta(z)\,. \tag{84}$$

With $\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}$ we obtain

$$\begin{aligned}\boldsymbol{A}(t, \boldsymbol{r}) &= \frac{e^{-i\omega t + ikr}}{4\pi r}R(I_0/c)\pi\,(-\hat{\boldsymbol{x}})\,, &(85)\\ &= \frac{e^{-i\omega t + ikr}}{4\pi r}\frac{-i\omega}{c}\boldsymbol{p} &(86)\end{aligned}$$

b) Then

$$\begin{aligned}\boldsymbol{B} &= \nabla \times \boldsymbol{A}\,, &(87)\\ &= \boldsymbol{n} \times \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}(t, \boldsymbol{r})\,, &(88)\\ &= \frac{e^{-i\omega t + ikr}}{4\pi r}(\boldsymbol{n} \times -\hat{\boldsymbol{x}})(-ikR)(I_0/c) &(89)\\ &= \frac{e^{-i\omega t + ikr}}{4\pi r}\cos\theta(-\hat{\boldsymbol{y}})(-ik\pi R)(I_0/c) &(90)\end{aligned}$$

c) The radiated power is

$$\frac{dP}{d\Omega} = \frac{c}{2}\mathrm{Re}(r^2\boldsymbol{n} \cdot (\boldsymbol{E} \times \boldsymbol{B}^*))\,. \tag{91}$$

With $\boldsymbol{E} = -\boldsymbol{n} \times \boldsymbol{B}$, we have

$$\boldsymbol{n} \cdot (-\boldsymbol{n} \times \boldsymbol{B}) \times \boldsymbol{B}^* = |\boldsymbol{B}|^2\,, \tag{92}$$

and

$$\frac{dP}{d\Omega} = \frac{c}{2} r^2 |\boldsymbol{B}|^2 \tag{93}$$

$$= \frac{c}{32\pi^2} \cos^2\theta \, (\pi k R I_0/c)^2 \tag{94}$$

It is perhaps useful to convert to MKS units:

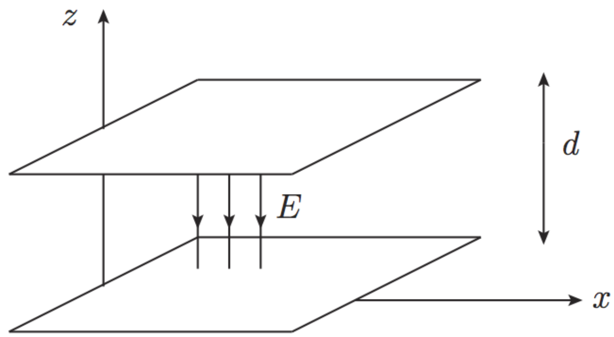
$$\frac{I_0}{c} \rightarrow \sqrt{\mu_o} I \tag{95}$$

$$c \rightarrow \frac{1}{\sqrt{\mu_o \epsilon_o}} \tag{96}$$

and using $\sqrt{\mu_o/\epsilon_o} = 376\,\mathrm{Ohm}$ we find

$$\frac{dP}{d\Omega} = 376\,\mathrm{Watts} \left(\frac{I_0}{\mathrm{amps}} \right)^2 \frac{(kR)^2}{32} \cos^2\theta \tag{97}$$

3. Since the magnetic field is in the $-\hat{\boldsymbol{y}}$ direction, for light propagating along the z axis the electric field is in the $-\hat{\boldsymbol{x}}$ direction, *i.e.* along the direction of the dipole moment.



5 Physics of the relativistic stress tensor

Consider a capacitor at rest. The area of each plate is A, and the electric field between the plates is E. The plates are orthogonal to the $x-$axis (see figure). The rest mass of each plate is $M_{\rm pl}$. The plates are kept a distance d apart by four thin columns (not shown). We assume that each of these columns have mass $M_{\rm col}$, and there is a stress tensor in the columns due to the electric attraction of the plates. (There is also a surface stress tensor in the plates due to the electric repulsion of the charges on the plates, but you won't need this.)

1. Write down the expression for the energy-momentum tensor of the electromagnetic field $\Theta^{\mu\nu}_{\rm em}$ in terms of the Maxwell field strength $F^{\mu\nu}$. Show that the total rest mass $Mc^2 = \int {\rm d}^3r\, \Theta^{00}_{\rm tot}$ of the capacitor setup is:

$$M_{\rm tot}c^2 = 2M_{\rm pl}c^2 + 4M_{\rm col}c^2 + \frac{1}{2}E^2Ad \tag{98}$$

   Remark. In practice the field term is very small compared to the first two terms, but we will include its effect in this problem.

2. Determine the non-vanishing components of the electromagnetic stress tensor integrated over space:

$$\int {\rm d}^3r\, \Theta^{\alpha\beta}_{\rm em}. \tag{99}$$

   (Hints: $\int \Theta^{xx}_{\rm em}$, $\int \Theta^{yy}_{\rm em}$, $\int \Theta^{zz}_{\rm em}$, $\int \Theta^{00}_{\rm em}$ are non-zero.)

3. Show that for a stationary configuration that

$$\int {\rm d}^3r\, \Theta^{ij}_{\rm tot}(\boldsymbol{r}) = 0 \tag{100}$$

   (Hints: Explain why $\partial_k\Theta^{kj}_{\rm tot} = 0$, and then study the expression $\partial_k(x^i\Theta^{kj}_{\rm tot})$)

4. Determine $\int_{\rm col} \Theta^{zz}_{\rm mech}$ in the columns, and interpret your result physically by showing the forces involved with a free body diagram.

5. Consider now an observer in frame K who is moving in the positive $z-$direction with velocity v relative to the rest frame of the capacitor. According to special relativity the energy of the capacitor in frame K is γMc^2 where $\gamma = (1-(v/c)^2)^{1/2}$.

(a) Show that the integrated electromagnetic stress tensor in frame K, $\underline{\Theta}^{00}_{\mathrm{em}}$, is

$$\int d^3\underline{r}\, \underline{\Theta}^{00}_{\mathrm{em}}(\underline{r}) = \frac{1}{2} E^2 A d \sqrt{1-(v/c)^2} \tag{101}$$

Here $\underline{r}$ are the boosted coordinates.

(b) Show that the integrated mechanical stress tensor including the plates and the columns

$$\int d^3\underline{r}\, \underline{\Theta}^{00}_{\mathrm{mech}}(\underline{r}) = \gamma\,(2M_{\mathrm{pl}}c^2 + 4M_{\mathrm{col}}c^2) + \frac{1}{2} E^2 A d\, \frac{(v/c)^2}{\sqrt{1-(v/c)^2}} \tag{102}$$

(c) Use these results to compute

$$\int d^3\underline{r}\, \underline{\Theta}^{00}_{\mathrm{tot}}(\underline{r}) \tag{103}$$

in frame K and comment on the simple result.