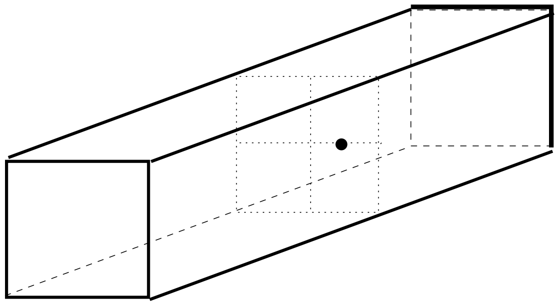
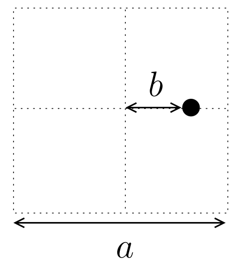
1 A charge in a rectangular tube

Consider a point charge placed in an infinitely long grounded rectangular tube as shown below. The sides of the square cross sectional area of the tube have length a.





1. (2 points) Show that the solutions to the *homogeneous* Laplace equation (i.e. without the extra point charge) are linear combinations of functions of the form

$$\Phi(k_x z)\, \Phi(k_y y)\, e^{\pm \kappa_z z} \qquad \text{where} \qquad \Phi(u) = \begin{cases} \cos(u) \text{ or } \sin(u) \end{cases} \tag{1}$$

   for specific values of k_x, k_y and κ_z. Determine the allowed the values of k_x, k_y and κ_z and their associated functions.

2. (4 points) Now consider a point charge displaced from the center of the tube by a distance b in the x direction, i.e. the coordinatess of the charge are $\boldsymbol{r}_o = (x, y, z) = (b, 0, 0)$. Use the method of images to determine the potential.

3. (7 points) As an alternative to the method of images, use a series expansion in terms of the homogeneous solutions of part (a) to determine the potential from the point charge described in part (b).

4. (7 points) Determine the asymptotic form of the surface charge density, and the force per area on the walls of the rectangular tube far from the point charge, i.e. $z \gg a$.

Solution

1. The Laplace equation is
$$-\nabla^2\varphi = 0 \tag{2}$$
Separating variables with $\varphi = X(x)Y(y)Z(z)$ we must have
$$-\frac{d^2X}{dx^2} = k_x^2 X \tag{3a}$$
$$-\frac{d^2Y}{dy^2} = k_y^2 Y \tag{3b}$$
$$-\frac{d^2Z}{dz^2} = k_z^2 Z \tag{3c}$$
In order to satisfy Eq. (2), the separation constants satisfy
$$k_x^2 + k_y^2 + k_z^2 = 0 \tag{4}$$
and thus
$$\frac{d^2Z}{dz^2} = \kappa^2 Z \qquad \text{with} \qquad \kappa = \sqrt{k_x^2 + k_y^2} \tag{5}$$
The solutions to Eq. (3a) may be either sin or cos
$$X(x) = \Phi(k_x x)\,, \tag{6}$$
with k_x at this point still arbitrary. In order to satisfy the boundary conditions $X(\pm a/2) = 0$, we require for the cos functions that
$$k_x a/2 = (n + \tfrac{1}{2})\pi\,. \tag{7}$$
Similarly, for the sin functions
$$k_x a/2 = n\pi\,. \tag{8}$$
Thus, the general form is
$$X_n(x) = \Phi_n(k_n x) \qquad n = 0, 1, \ldots \tag{9}$$
with $k_n = (n+1)\pi/a$ and
$$\Phi_n(u) = \begin{cases} \cos(u) & \text{n even} \\ \sin(u) & \text{n odd} \end{cases}\,. \tag{10}$$
The $Y(y)$ direction follows by analogy
$$Y_m(y) \qquad \Phi_m(k_m x) \qquad m = 0, 1, \ldots \tag{11}$$
with $k_m = (m+1)\pi/a$ The solutions to the z direction are
$$Z(z) = e^{\pm\kappa z} \qquad \kappa = \sqrt{k_n^2 + k_m^2} \tag{12}$$

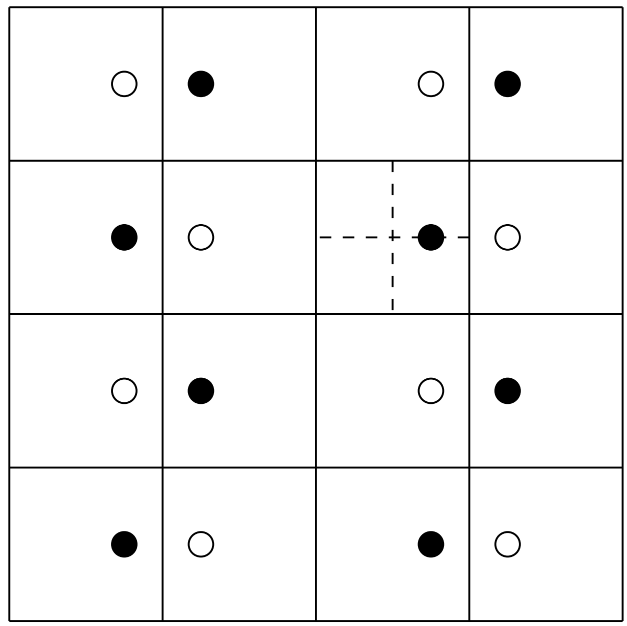


Figure 1: Arrangement of image charges. The black charges idicate plus charges, while the white charges indicate negative charges. The origin of coordinates is indicated with the dashed lines. The real charge is displaced by a distance b from the origin.

2. The image charges may be placed in a rectangular lattice as shown below. Their are four types of charges with coordinates

$$\boldsymbol{r}_1(n,m) = (b + 2na)\hat{\boldsymbol{x}} + 2ma\hat{\boldsymbol{y}} \tag{13}$$
$$\boldsymbol{r}_2(n,m) = ((2n+1)a - b)\hat{\boldsymbol{x}} + 2ma\hat{\boldsymbol{y}} \tag{14}$$
$$\boldsymbol{r}_3(n,m) = (b + 2na)\hat{\boldsymbol{x}} + (2m+1)a\hat{\boldsymbol{y}} \tag{15}$$
$$\boldsymbol{r}_4(n,m) = ((2n+1)a - b)\hat{\boldsymbol{x}} + (2m+1)a\hat{\boldsymbol{y}} \tag{16}$$

where n, m are integers. Then the potential is

$$\phi(\boldsymbol{r}) = \frac{q}{4\pi} \sum_{n,m=0}^{\infty} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_1(n,m)|} - \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_2(n,m)|} - \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_3(n,m)|} + \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_4(n,m)|} \tag{17}$$

3. For the potential at $\boldsymbol{r}$ due to a point charge at $\boldsymbol{r}_o = (b, 0, 0)$, we expand the potential as

$$\phi(\boldsymbol{r}; \boldsymbol{r}_o) = \left(\frac{2}{a}\right)^2 \sum_{n,m=0}^{\infty,\infty} X_n(x) X_n(b)\, Y_m(y) Y_m(0)\, g_{n,m}(z) \tag{18}$$

and substitute into the Poisson equation

$$-\nabla^2 \varphi(\boldsymbol{r}; \boldsymbol{r}_o) = q\delta(x - b)\delta(y)\delta(z)\,. \tag{19}$$

The leading factors $(2/a)^2$ arise from the fact that we have not normalized the eigenfunctions X and Y

$$\int_{-a/2}^{a/2} dx\, X_n(x) X_{n'}(x) = \frac{a}{2}\, \delta_{n,n'} \tag{20}$$

$$\int_{-a/2}^{a/2} dy\, Y_m(y) Y_{m'}(y) = \frac{a}{2}\, \delta_{m,m'} \tag{21}$$

If $g_{n,m}(z)$ satisfies

$$\left(k_n^2 + k_m^2 - \frac{\partial^2}{\partial z^2} \right) g_{n,m}(z) = q\delta(z)\,, \tag{22}$$

then using the completeness relation

$$\frac{2}{a} \sum_n X_n(x) X_n(x_o) = \delta(x - x_o) \tag{23}$$

$$\frac{2}{a} \sum_m Y_m(x) Y_m(x_o) = \delta(y - y_o) \tag{24}$$

it is not difficult to show that Eq. (19) is satisfied.

The solution to Eq. (22) is

$$g_{n,m}(z) = \begin{cases} Ae^{-\kappa_{n,m} z} & z > 0 \\ Ae^{\kappa_{n,m} z} & z < 0 \end{cases} \tag{25}$$

Integrating across the δ-fcn in Eq. (22) we have

$$- \left. \frac{dg}{dz} \right|_{z=0^+} + \left. \frac{dg}{dz} \right|_{0^-} = q \tag{26}$$

With this requirement $A = \frac{q}{2\kappa_{n,m}}$ and

$$\phi(\boldsymbol{r}; \boldsymbol{r}_o) = \frac{4q}{a^2} \sum_{n,m=0}^{\infty,\infty} X_n(x) X_n(b)\, Y_m(y) Y_m(0) \frac{e^{-\kappa_{n,m}|z|}}{2\kappa_{n,m}} \tag{27}$$

4. At asymptotic distances the terms with the smallest $\kappa_{n,m}$ dominate the sum. We then have only the contribution from $n = m = 0$ mode, and

$$\kappa_{0,0} = \sqrt{2}\pi/a\,. \tag{28}$$

The potential reads

$$\phi(\boldsymbol{r}; \boldsymbol{r}_o) \simeq \frac{4q}{a^2} \cos(\pi x/a) \cos(\pi b/a)\, \cos(\pi y/a) \frac{e^{-\kappa_{0,0}|z|}}{2\kappa_{0,0}} \tag{29}$$

or

$$\phi(\boldsymbol{r};\boldsymbol{r}_o) \simeq \frac{\sqrt{2}q}{\pi a} cos(\pi x/a)\cos(\pi b/a)\ \cos(\pi y/a) e^{-\sqrt{2}\pi|z|/a} \tag{30}$$

Let us calculate the charge density on the bottom plate

$$\sigma = \boldsymbol{n}\cdot\boldsymbol{E} = -\partial_y \phi\big|_{y=-a/2}\ , \tag{31}$$

$$= -\ \frac{\sqrt{2}q}{a^2}\cos(\pi x/a)\cos(\pi b/a)\, e^{-\sqrt{2}\pi|z|/a}\,. \tag{32}$$

Finally, the force per area on the bottom plate is

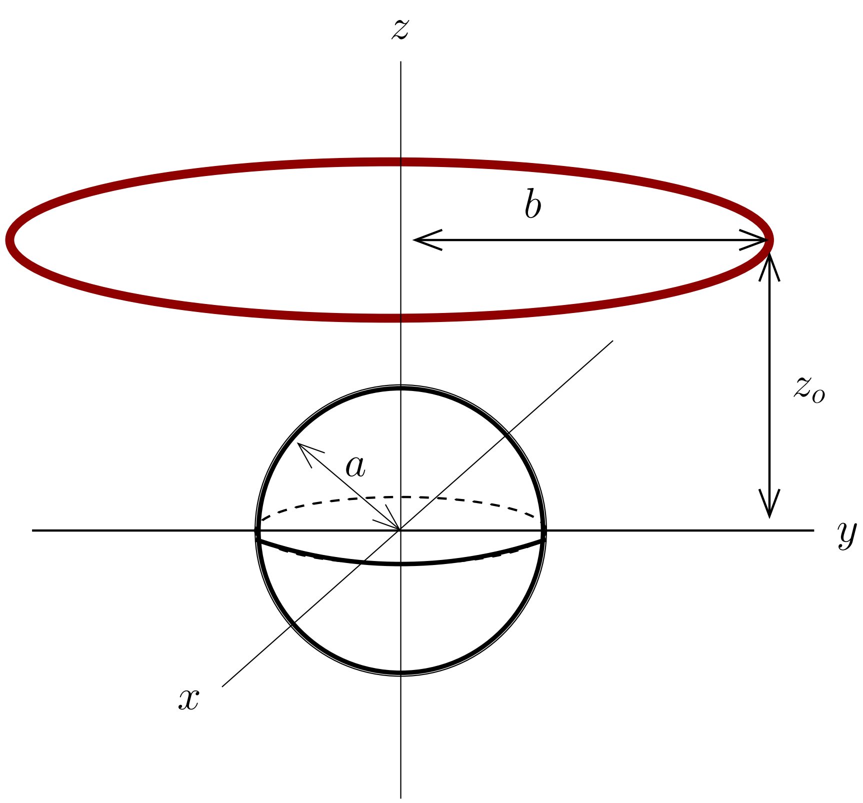
$$\frac{F^y}{A} = \frac{\sigma^2}{2}\ , \tag{33}$$

$$= \frac{q^2}{a^4}\cos^2(\pi x/a)\cos^2(\pi b/a)\, e^{-2\sqrt{2}\pi|z|/a}\,. \tag{34}$$

The direction of the force is into the tube. The other walls of the tube have the same force per area.

2 A ring and a sphere in a magnetic field

A sphere of radius a with magnetic permeability μ is placed in an external slowly varying (homogeneous) magnetic field, $\boldsymbol{B}_{\rm ext}(t) = B_o(t)\,\hat{\boldsymbol{z}} = \mathcal{B}\cos(\omega t)\hat{\boldsymbol{z}}$. Placed above the sphere at height z_o is an ohmic ring of radius b and resistance $\mathcal{R}$. The center of the ring coincides with the z-axis and the plane of the ring points along the z-axis (see below).



(a) (6 points) The induced magnetic moment of the sphere is proporitonal to the external field

$$\boldsymbol{m} = \alpha_B \boldsymbol{B}_{\rm ext}\,. \tag{35}$$

Determine the polarizability, α_B. Neglect the fields from the currents induced in the ring.

(Hint: recall that for a permeable sphere in a constant external magnetic field, the magnetic field outside the sphere is that of an induced magnetic dipole plus the external field, while the magnetic field inside the sphere is constant, $\boldsymbol{B}_{\rm in} = B_{\rm in}\,\hat{\boldsymbol{z}}$. Determine α_B and $B_{\rm in}$ from the appropriate boundary conditions at the surface of the sphere.)

(b) (6 points) Determine the current induced in the ring.

(c) (2 points) Under what conditions can the induced magnetic fields from the ring be neglected in part (a)?

(d) (6 points) Determine the force on the ring.

Solution

(a) The boundary conditions read

$$\boldsymbol{n} \times (\boldsymbol{H}_{\rm out} - \boldsymbol{H}_{\rm in}) = 0 \tag{36}$$
$$\boldsymbol{n} \cdot (\boldsymbol{B}_{\rm out} - \boldsymbol{B}_{\rm in}) = 0 \tag{37}$$

In terms of components

$$H_{\theta,{\rm out}} - H_{\theta,{\rm in}} = 0 \tag{38}$$
$$B_{r,{\rm out}} - H_{r,{\rm in}} = 0 \tag{39}$$

With the magnetic field of a dipole

$$\boldsymbol{B}_{\rm out} = B_o\, \hat{\boldsymbol{z}} + \frac{3\hat{\boldsymbol{r}}(\hat{\boldsymbol{r}} \cdot \boldsymbol{m}) - \boldsymbol{m}}{4\pi r^2} \tag{40}$$
$$\boldsymbol{B}_{\rm in} = B_o\, \hat{\boldsymbol{z}} + \frac{3\hat{\boldsymbol{r}}(\hat{\boldsymbol{r}} \cdot \boldsymbol{m}) - \boldsymbol{m}}{4\pi r^2} \tag{41}$$

we see that

$$B_{r,{\rm out}} = \frac{2m\cos\theta}{4\pi a^3} + B_o\cos\theta \tag{42}$$
$$H_{\theta,{\rm out}} = \frac{m\sin\theta}{4\pi a^3} - B_o\sin\theta \tag{43}$$

Inside we have

$$B_{r\,{\rm in}} = B_{\rm in}\cos\theta \tag{44}$$
$$H_{\theta\,{\rm in}} = -\frac{1}{\mu}B_{\rm in}\sin\theta \tag{45}$$

Putting together the ingredients we have

$$\frac{m}{4\pi a^3} - B_o + \frac{B_{\rm in}}{\mu} = 0 \tag{46}$$
$$\frac{2m}{4\pi a^3} + B_o - B_{\rm in} = 0 \tag{47}$$

Solving these equation for m and $B_{\rm in}$ we get

$$m = B_o(4\pi a^3)\frac{\mu - 1}{2+\mu} \tag{48}$$
$$B_{\rm in} = B_o\frac{3\mu}{2+\mu} \tag{49}$$

(b) The flux through the loop has two contributions: the external magnetic field and the induced dipole. The external dipole contribution is simply

$$\Phi_{B,{\rm ext}} = B_o(t)\pi b^2\,. \tag{50}$$

The dipole contribution is most easily found using the vector potential

$$\Phi_{B,\mathrm{dip}} = \int \boldsymbol{B}\cdot d\boldsymbol{a} = \oint \boldsymbol{A}\cdot d\boldsymbol{\ell}\,. \tag{51}$$

With the vector potential of the dipole

$$\boldsymbol{A} = \frac{\boldsymbol{m}\times\hat{\boldsymbol{r}}}{4\pi r^2} \tag{52}$$

we have

$$A_\phi = \frac{m\sin\theta}{4\pi(z^2+b^2)} \tag{53}$$

So with $\sin\theta = b/\sqrt{z^2+b^2}$ we have

$$\Phi_{B,\mathrm{dip}} = \frac{m(t)}{2}\frac{b^2}{(z^2+b^2)^{3/2}} \tag{54}$$

$$= \alpha_B\frac{B_o(t)}{2}\frac{b^2}{(z^2+b^2)^{3/2}} \tag{55}$$

Thus the magnetic current is

$$I(t) = -\frac{1}{c\mathcal{R}}\partial_t\Phi_B(t) \tag{56}$$

Or

$$I(t) = \frac{-\dot{B}_o(t)\pi b^2}{c\mathcal{R}}\left[1 + \frac{\alpha_B}{2\pi}\frac{1}{(z^2+b^2)^{3/2}}\right] \tag{57}$$

(c) The current in the loop produces a field at the sphere of order $I(t)/[c(b^2+z^2)^{1/2}]$. We should compare this field to B_o, yielding the condition:

$$\frac{\omega B_o\pi b^2}{c^2\mathcal{R}}\frac{1}{(z^2+b^2)^{1/2}} \ll B_o\,. \tag{58}$$

Taking b and z the same order of magnitude $b\sim z$ as drawn in the figure,

$$\frac{\omega\pi b}{2\pi c^2\mathcal{R}} \ll 1\,. \tag{59}$$

This is the answer.

It is useful to interpret the answer. The resistance is $\mathcal{R} = 2\pi b/(\sigma A)$ where A is the cross section of the wire and σ is the conductivity, yielding

$$\frac{\omega\sigma}{4\pi c^2}A \ll 1\,. \tag{60}$$

Recognizing the magnetic diffusion coefficient $D{=}c^2/\sigma$ of the wire and the skin depth $\delta(\omega)\sim\sqrt{D/\omega}$, we rewrite the condition as

$$\frac{A}{\pi\delta^2(\omega)} \ll 1\,. \tag{61}$$

(d) For the force we have the contribution of the constant field B_o and the field of the sphere $B_{\rm dip}$.

Using the right hand rule we see that the constant field produces no net force. All the forces of from the static field lie in the plane of the loop, tending to deform the ring but providing no net force.

From the dipole we have the Lorentz force

$$F^z = \int b d\phi \, \frac{I(t)}{c} \, \hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{\phi}} \times \boldsymbol{B}_{\rm dip}) \, . \tag{62}$$

With the diople field,

$$\boldsymbol{B}_{\rm dip} = \frac{3\hat{\boldsymbol{r}} \cdot (\hat{\boldsymbol{r}} \cdot \boldsymbol{m}) - \boldsymbol{m}}{4\pi r^3} , \tag{63}$$

the magnetic moment $m(t) = \alpha_B B_o(t) \hat{\boldsymbol{z}}$, the cross products

$$\hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{r}}) = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\theta}} = -\sin\theta \, , \tag{64}$$

$$\hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{z}}) = 0 \, , \tag{65}$$

we find

$$F^z = -\int b d\phi \, I(t)/c \, \frac{3 \sin\theta \cos\theta \, m(t)}{4\pi (z_o^2 + b^2)^{3/2}} \tag{66}$$

Thus

$$F^z = \left(-\frac{I(t) B_o(t) b}{c} \right) \frac{3}{4} \frac{\sin(2\theta) \alpha_B}{(z_o^2 + b^2)^{3/2}} \tag{67}$$

This is the answer after substituting the results of part (b).

After minor manipulations we find

$$F^z = \left(\frac{dB_o^2(t)}{dt} \frac{\pi b^3}{c^2 \mathcal{R}} \right) \frac{3}{8} \frac{\sin(2\theta) \, \alpha_B}{(z_o^2 + b^2)^{3/2}} \left[1 + \frac{\alpha_B}{2\pi (z_o^2 + b^2)^{3/2}} \right] \tag{68}$$

3 Waves in Metals

Consider an ohmic metal with high (but not infinite) conductivity σ and magnetic permeability[1] $\mu = 1$, so that $\boldsymbol{B} = \boldsymbol{H}$.

1. (6 pnts) Show that for harmonic time dependence, and high conductivity[2] $\sigma \gg \omega$, that damped wave like solutions propagating in z-direction in the metal take the approximate form:

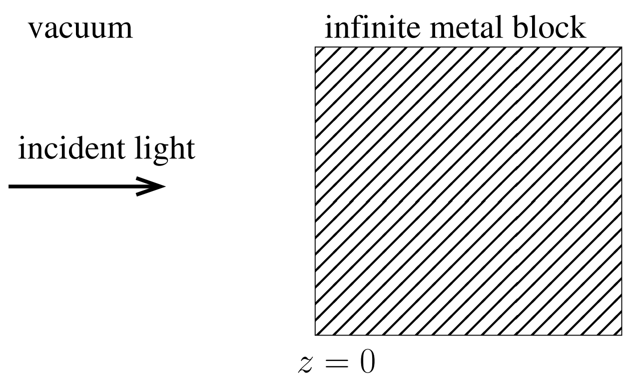
$$\boldsymbol{H}(t, z) = \boldsymbol{H}_c e^{-i\omega t + ik_c z} \tag{69}$$

where[3]

$$k_c = \frac{1+i}{\sqrt{2}} \frac{\sqrt{\sigma\omega}}{c} \tag{70}$$

2. (4 pnts) The electric field obeys a similar equation, $\boldsymbol{E}(t, z) = \boldsymbol{E}_c e^{-i\omega t + ik_c z}$. Use the Maxwell equations to express the amplitude of the electric field $\boldsymbol{E}_c$ in terms of the magnetic field $\boldsymbol{H}_c$.

3. (4 pnts) Now consider a linearly polarized plane wave in vacuum of frequency ω, which is normally incident upon a semi-infinite metal block with *infinite* conductivity as shown below.



When the metal has infinite conductivity, the amplitude of the reflected equals equals the amplitude of the incident wave, but the polarization of the reflected wave is inverted. Explain this familiar fact using the appropriate boundary conditions.

4. (6 pnts) Now consider the same reflection problem as in part 3, but this time the metal has a large (but finite) conductivity σ. Determine the electric and magnetic fields in the metal to leading order in ω/σ. The amplitude of the incident wave is E_o.

5. (not part of exam). Determine the energy lost into the metal in terms of the input magnetic field. (See lecture for two different ways to do this).

[1] In SI units this reads $\mu = \mu_o$

[2] In SI units this condition reads $(\sigma/\epsilon_o) \gg \omega$

[3] This is written in Heaviside-Lorentz units. In SI units $k_c = (1+i)/\sqrt{2}\,\sqrt{\omega(\sigma/\epsilon_o)}\,/c$, while in Gaussian units, $k_c = (1+i)/\sqrt{2}\,\sqrt{4\pi\sigma\omega}\,/c$.

Problem: (8.1 Induction and the energy in static Magnetic fields)

Consider a closed circuit of wire formed into a circular coil of n turns with radius a, resistance R, and self-inductance L. The coil rotates around the z-axis in a uniform magnetic field H directed along the x-axis (see below).

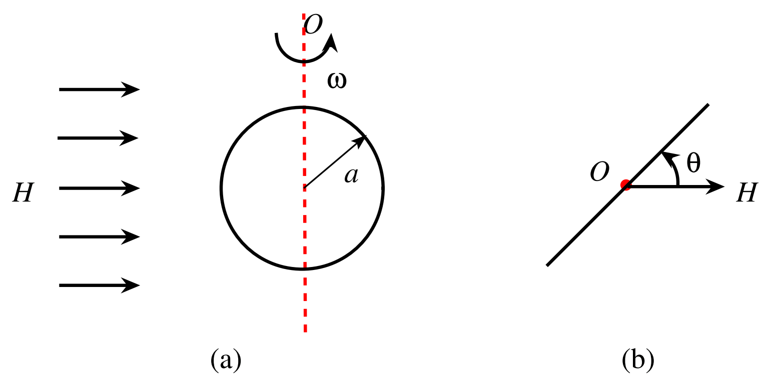


Figure 2: (a) side view; (b) top view.

a) (6 points) Find the current in the coil as a function θ for rotation at a constant angular velocity ω. Here $\theta(t) = \omega t$ is the angle between the plane of the coil and H (the x-axis).

b) (4 points) Find the externally applied torque that is needed to maintain the coil's uniform rotation.

Note: in all parts you should assume that all transient effects have died away.

Solution:

a) Let I be the current in the coil, we have

$$\oint_{\text{coil}} E \cdot dr = IR = -L\frac{dI}{dt} - \frac{1}{c}\frac{\partial \Phi_H}{\partial t}\,, \tag{71}$$

where the flux is given by $\Phi_H = \pi a^2\, n\, H \sin\theta(t)$ with $\theta(t) = \omega t$. With these phase conventions, the area vector of the loop points in the negative $\hat{\boldsymbol{y}}$ direction at $t = 0$ and in the $\hat{\boldsymbol{x}}$ direction at $\omega t = \pi/2$. Thus the circulation of a positive current at $t = 0$ is specified with the right hand rule with the thumb pointing in the negative $\hat{\boldsymbol{y}}$ direction.

From Eq. (71), we have the differential equation for the current,

$$L\,\frac{dI}{dt} + R\,I = -\frac{\pi a^2}{c} n\, H\,\omega\, \cos(\omega t)\,. \tag{72}$$

We will write this as

$$L\,\frac{dI}{dt} + R\,I = -\frac{\pi a^2}{c} n\, H\,\omega\, e^{-i\omega t}\,. \tag{73}$$

with the understanding that one is supposed to take the real part. Taking a trial solution $I(t) = I_\omega e^{-i\omega t}$, we solve for I_ω and find

$$I_\omega = \frac{\pi a^2\, n\, H\,\omega}{c}\frac{1}{R - i\omega L}\,. \tag{74}$$

Thus

$$\begin{aligned} I(t) &= -\frac{\pi a^2\, n\, H\,\omega}{c}\,\frac{1}{2}\left[\frac{e^{i\omega t}}{R + i\omega L} + \frac{e^{-i\omega t}}{R - i\omega L}\right] \\ &= -\frac{\pi a^2\, n\, H}{c}\frac{\omega}{\sqrt{R^2 + \omega^2 L^2}}\, \cos(\omega t + \phi)\,, \end{aligned} \tag{75}$$

where the phase $\phi = \tan^{-1}(-\omega L/R)$.

b) The rotating coil has a magnetic dipole moment, $\boldsymbol{\mu}(t) = I(t)\vec{A}(t)/c$. With the conventions of the previous part we have

$$\boldsymbol{\mu}(t) = m_o \cos(\omega t + \phi)\, \left(-\sin(\omega t)\hat{\boldsymbol{x}} + \cos(\omega t)\hat{\boldsymbol{y}}\right)\,. \tag{76}$$

where

$$m_o \equiv \left(\frac{\pi a^2 n}{c}\right)^2 \frac{\omega}{\sqrt{R^2 + \omega^2 L^2}} H\,. \tag{77}$$

The torque on the loop is $\boldsymbol{\mu} \times \mathbf{H}$, and an external torque of $\boldsymbol{\tau}_{\text{ext}} = -\boldsymbol{\mu} \times \mathbf{H}$ is needed to keep the coil rotating at a constant angular velocity is (with $\mathbf{H} = H\hat{\boldsymbol{x}}$) :

$$\boldsymbol{\tau}_{\text{ext}}(t) \quad = \quad m_o H \cos(\omega t + \phi) \cos(\omega t)\, \hat{\boldsymbol{z}} \tag{78}$$

4 (Part of) A time dependent dipole

Consider an electric dipole at the spatial origin ($\boldsymbol{x} = 0$) with a time dependent electric dipole moment oriented along the z-axis, *i.e.*

$$\boldsymbol{p}(t) = p_o \cos(\omega t) \hat{\boldsymbol{z}} \,, \tag{79}$$

where $\hat{\boldsymbol{z}}$ is a unit vector in the z direction.

1. Recall that the near and far fields of the time dependent dipole are qualitatively different. Estimate the length scale that separates the near and far fields.

2. In the near field regime, *estimate* how the electric and magnetic field strengths decrease with the radius r. (r is the distance from the origin to the observation point.)

3. Using a system of units where $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same units (such as Gaussian or Heaviside-Lorentz), *estimate* the ratio E/B at a distance r in the near field[4]. Is this ratio large or small?

4. Determine the electric and magnetic fields to the lowest non-trivial order in the near field (or quasi-static) approximation.

[4] In SI units this question reads, "Estimate the ratio E/cB at a distance r in the near field."

Solution

1. The speed of light and the frequency define a length scale

$$1/(R_o) = \omega/c$$

For distances less than R_o a quasi-static approximation may be used. For distances greater than R_o the finite speed of light must be considered to calculate the radiation fields

2. There are various ways to do this. Perhaps the most direct is to use the gauge potentials in the lorentz gauge. We will not do this, but use the Maxwell equations directly.

The electric field in the near field region is just the field of a dipole

$$\boldsymbol{E} = \frac{1}{4\pi r^3}\left[3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}\right] \tag{80}$$

Clearly $\boldsymbol{E}$ lies in $\hat{r}, \hat{\theta}$ plane. So

$$\boldsymbol{E} = \frac{1}{4\pi r^3}\left[(2p_o(t)\cos\theta)\,\hat{\boldsymbol{r}} + (p_o(t)\sin\theta)\,\hat{\boldsymbol{\theta}}\right] \tag{81}$$

where $p_o(t) = p_o\cos(\omega t)$

Since

$$\nabla\times\boldsymbol{B} = \frac{1}{c}\partial_t\boldsymbol{E} \tag{82}$$

We try $\boldsymbol{B}$ in the ϕ direction, with $B_\phi(r,\theta)$. Then

$$(\nabla\times\boldsymbol{B})_\theta = -\frac{1}{r}\partial_r(rB_\phi) = \frac{1}{4\pi r^3}(\partial_t p_o)\sin\theta \tag{83}$$

Integrating with respect to r we find

$$B_\phi = \frac{1}{4\pi r^2 c}(\partial_t p_o)\sin\theta + \frac{f(\theta)/R_o^2}{r} \tag{84}$$

Where $f(\theta)$ is a dimensionless integration constant, and we have inserted factors of R_o to make up the dimensions. The terms proportional to $1/r$ can be dropped in the near field regime since it is smaller by r/R_o than the $\frac{1}{r^2}$ term. Thus

$$B_\phi = \frac{1}{4\pi r^2 c}(\partial_t p_o)\sin\theta\,. \tag{85}$$

Then one verifies that

$$(\nabla\times\boldsymbol{B})_r = \frac{1}{r\sin\theta}\partial_\theta(\sin\theta B_\phi) = \frac{1}{4\pi r^3 c}(\partial_t p_o)\,2\cos\theta = \frac{1}{c}\partial_t E_r \tag{86}$$

showing that B_ϕ satisfies the Maxwell equations.

Another way to do this is by recognizing a formal simlarity to the magnetic dipole. The vector potential of a magnetic dipole satisfies

$$\nabla \times \boldsymbol{A} = \boldsymbol{B} \text{ of a dipole} = \frac{3\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{m}) - \boldsymbol{m}}{4\pi r^3} \tag{87}$$

and equals

$$\boldsymbol{A} = \frac{\boldsymbol{m}\times\hat{\boldsymbol{r}}}{4\pi r^2}\,. \tag{88}$$

Here we are trying to solve

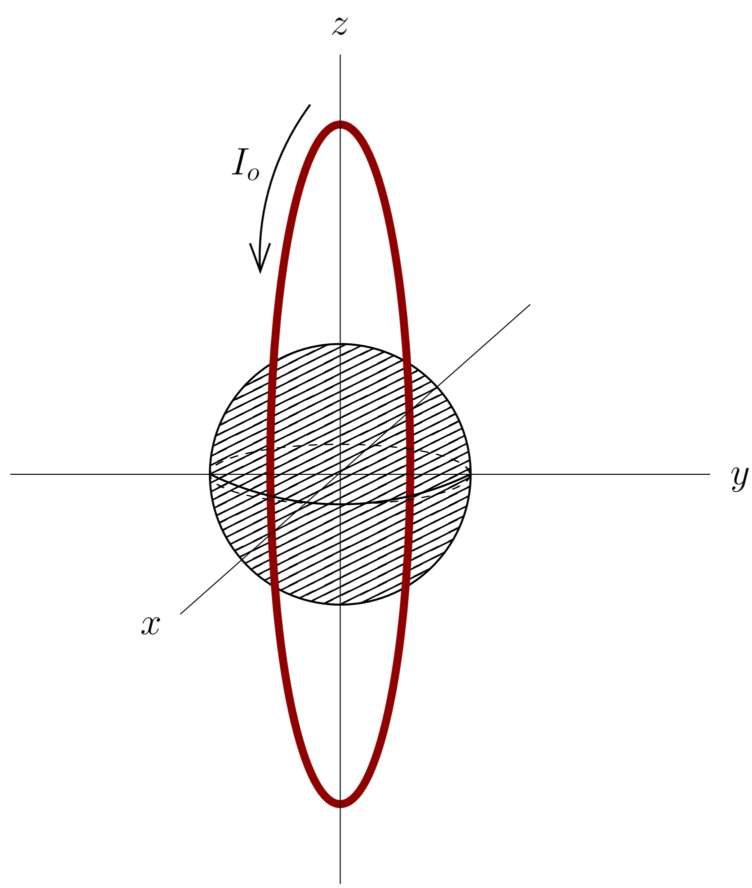
$$\nabla \times \boldsymbol{B} = \frac{3\boldsymbol{n}(\boldsymbol{n}\cdot\dot{\boldsymbol{p}}(t)/c) - \dot{\boldsymbol{p}}(t)/c}{4\pi r^3}\,. \tag{89}$$

So we have (by analogy with the magnetic dipole)

$$\boldsymbol{B} = \frac{\dot{\boldsymbol{p}}(t)/c\times\hat{\boldsymbol{r}}}{4\pi r^2} = \frac{1}{4\pi r^2 c}(\partial_t p_o)\sin\theta\,\widehat{\boldsymbol{\phi}} \tag{90}$$

5 A magnetized sphere and a circular hoop

A uniformly magnetized sphere of radius a centered at origin has a permanent total magnetic moment $\mathbf{m} = m\,\hat{\boldsymbol{z}}$ pointed along the z-axis (see below). A circular hoop of wire of radius b lies in the xz plane and is also centered at the origin. The hoop circles the sphere as shown below, and carries a small current I_o (which does not appreciably change the magnetic field). The direction of the current I_o is indicated in the figure.



1. Determine the magnetic field $\boldsymbol{B}$ inside and outside the magnetized sphere.
2. Determine the bound surface current on the surface of the sphere.
3. What is the direction of the net-torque on the circular hoop? Indicate on the figure how the circular hoop will tend to rotate and explain your result.
4. Compute the net-torque on the circular hoop.

Solution

1. The magnetic field outside is one of a magnetic dipole, where all of magnetic moment is placed at the origin
$$\boldsymbol{B} = \frac{1}{4\pi r^3}\left[3(\mathbf{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \mathbf{m}\right] \tag{91}$$
Inside sphere, the magnetic field is constant
$$\boldsymbol{B} = B_o\,\hat{\boldsymbol{z}} \tag{92}$$
The constant B_o can be picked off from the boundary conditions.
The boundary conditions read
$$\boldsymbol{n}\times(\boldsymbol{B}_2 - \boldsymbol{B}_1) = \frac{\boldsymbol{K}_b}{c} \tag{93}$$
$$\boldsymbol{n}\cdot(\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{94}$$
Then from the boundary conditions at $r = a$
$$B_r|_{\rm out} = B_r|_{\rm in}\ . \tag{95}$$
With the magnetic field outside the sphere
$$B_r|_{\rm out} = \frac{1}{4\pi r^3}2m\cos\theta\,, \tag{96}$$
and inside the sphere
$$\hat{\boldsymbol{r}}\cdot\boldsymbol{B}|_{\rm in} = B_o\,\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{z}} = B_o\cos\theta\,, \tag{97}$$
comparison at $r = a$ gvies
$$B_o = \frac{1}{4\pi a^3}2m\,. \tag{98}$$
For later reference we note that with $M = m/(4\pi a^3/3)$
$$H_o = B_o - M = -\frac{m}{4\pi a^3} \tag{99}$$

2. The surface current is in the azimuthal direction
$$\boldsymbol{K} = K_o\,\hat{\boldsymbol{\phi}} \tag{100}$$
Inside we have
$$\boldsymbol{B} = B_o\hat{\boldsymbol{z}} = B_o\cos\theta\,\hat{\boldsymbol{r}} - B_o\sin\theta\,\hat{\boldsymbol{\theta}}\,, \tag{101}$$
while outside we have
$$\boldsymbol{B} = \frac{1}{4\pi r^3}2m\cos\theta\,\hat{\boldsymbol{r}} + \frac{1}{4\pi r^3}m\sin\theta\,\hat{\boldsymbol{\theta}}\,. \tag{102}$$
Then the jump condition reads
$$B_{\theta,\rm out} - B_{\theta,\rm in} = \frac{K_o}{c}\,. \tag{103}$$

Thus

$$K_o = c\left(\frac{1}{4\pi a^3}m + B_o\right)\sin\theta = \frac{3c}{4\pi a^3}m\sin\theta \tag{104}$$

One can verify using eq. (99)

$$H_{\theta,\mathrm{out}} - H_{\theta,\mathrm{in}} = \left(\frac{1}{4\pi r^3}m\sin\theta + H_o\sin\theta\right) = 0 \tag{105}$$

as should be the case since H is continuous in the absence of external macroscopic currents.

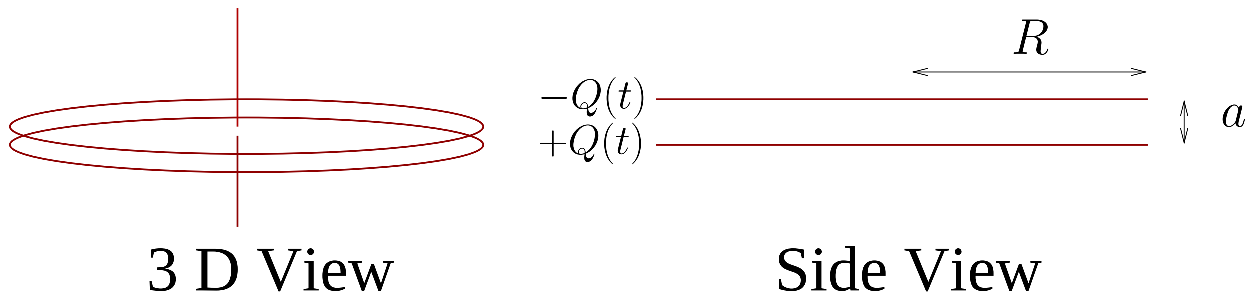
3. To compute the torque we first compute the lorentz force on a element of length $d\ell = bd\theta$.

$$\begin{aligned} dF =& \frac{I_o}{c}d\ell B_\perp & (106)\\ =& \frac{I_o}{c}bd\theta\ B_r & (107)\\ =& \frac{I_o}{c}bd\theta\ \frac{2m\cos\theta}{4\pi b^3} & (108)\end{aligned}$$

The right hand rule indicates that the force is in the $-\hat{\boldsymbol{y}}$ direction in the upper hemisphere, and in the positive $\hat{\boldsymbol{y}}$ direction in the lower hemisphere. This implies that the net torque points along the x-axis. This can be intuited by noting that the magnetic moment of the hoop tends to align with the magnetic field from the sphere

4. The torque around the x-axis

$$\begin{aligned} \tau =& \int d\tau = \int b\cos\theta\, dF & (109)\\ =& 2\int_0^\pi b\cos\theta\ \frac{I_o}{c}bd\theta\ \frac{2m\cos\theta}{4\pi b^3} & (110)\\ =& \frac{4m(I_o/c)b^2}{4\pi b^3}\int_0^\pi d\theta\cos^2\theta & (111)\\ =& \frac{4m(I_o/c)b^2}{4\pi b^3}\frac{\pi}{2} & (112)\\ =& \frac{2m}{4\pi b^3}\left[\frac{I_o}{c}\pi b^2\right] & (113)\end{aligned}$$



6 A circular capicitor

A circular capacitor of radius R and separation a, with $a \ll R$, is charged with a slow sinusoidal current, *i.e.* the charge on the plates is $Q(t) = \pm Q_o \sin(\omega t)$ as illustrated above. Neglect any fringing of the fields.

1. Determine the electric and magnetic fields in between the plates in a quasi-static approximation. Draw a picture to indicate the directions of the fields while the charge on the bottom plate is positive and increasing.

2. What are the size of typical corrections to the fields computed in part (1) due to the finite speed of light?

3. Write down the Maxwell equations for the gauge potentials ϕ and $\mathbf{A}$ in the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$

4. Determine the gauge potentials $(\phi, \mathbf{A})$ associated with the fields of part (1) and show that that they satisfy the Maxwell equations found in part (3) to the required order.

   This problem uses a different notation from the class. The curl of a vector field $\mathbf{F}$ in cylindrical coordinates is with $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$

$$\nabla \times \mathbf{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \hat{\mathbf{r}} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) \hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \hat{\mathbf{z}} \tag{114}$$

   The Laplacian is

$$\Delta^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r} \right) + \frac{\partial^2 f}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \tag{115}$$

Solution – Heavy Side Lorentz Units

1. The electric field is

$$\nabla \cdot \mathbf{E} = \rho \qquad E^z = \frac{Q(t)}{A}\hat{\mathbf{z}} \tag{116}$$

The magnetic field is determined from Amperes law with no current

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} - \nabla \times \mathbf{B} = 0 \tag{117}$$

So

$$B^\theta(2\pi r) = \frac{1}{c}\pi r^2 \partial_t E^z \tag{118}$$

Or

$$B^\theta = \frac{r\omega}{2c}\frac{Q}{A}\cos(\omega t) \tag{119}$$

2. Corrections are of order

$$\left(\frac{R\omega}{c}\right)^2 \tag{120}$$

3. Then we have

$$-\Box\varphi - \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t\phi + \nabla\cdot\boldsymbol{A}\right) =\rho \tag{121}$$

$$-\Box\boldsymbol{A} + \partial_i\left(\frac{1}{c}\partial_t\phi + \nabla\cdot\boldsymbol{A}\right) = \boldsymbol{j}/c \tag{122}$$

Taking $\rho = 0$ and $\boldsymbol{j} = 0$. Then taking the Coulomb gauge $\nabla \cdot A = 0$ we have

$$-\nabla^2\varphi =\rho \tag{123}$$

$$-\Box\boldsymbol{A} = -\,\partial_i\left(\frac{1}{c}\partial_t\phi\right) + \boldsymbol{j}/c \tag{124}$$

4. Solving the ϕ

$$-\,\partial_i\partial^i\phi = 0 \qquad \Longrightarrow \qquad \phi = -E^z(t)z \tag{125}$$

For $\mathbf{A}$ we have only a z component. And, we may drop ∂_t^2 in the quasi static approximation

$$-\frac{1}{c^2}\partial_t^2 A^z + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A^z}{\partial r}\right) = \frac{1}{c}\partial_t\partial^z\phi \tag{126}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A^z}{\partial r}\right) = \frac{-\omega}{c}\frac{Q}{A}\cos(\omega t) \tag{127}$$

Integrating this we find

$$A^z = -\frac{Q\omega}{4Ac}\cos(\omega t)r^2 \tag{128}$$

A straight forward sanity check gives $\mathbf{B} = \nabla \times A$

$$B^\theta = -\frac{\partial}{\partial r}A^z = \frac{Q}{A}\frac{r\omega_o}{2c}\cos\omega t \tag{129}$$