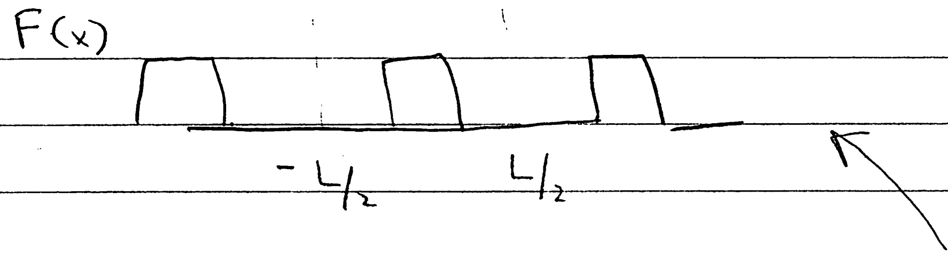
Complete Orthogonal Functions   fourier series + transform

The separation of the Laplace equations leads to a large number of complete eigenfunctions. These are usually <u>not normalized</u>. Here we will recall some properties of all eigen-fcn expansions.

Take Fourier Series for definiteness. The eigenfcns are:

$$\langle x|n\rangle = \psi_n(x) = \left[e^{ik_n x}\right] \quad \text{with} \quad k_n = \frac{2\pi n}{L}, \quad \text{then}$$



Take any periodic function, F(x), and expand in the eigen functions $\psi_n(x)$. We have the following properties of eigenfunctions

① <u>Orthogonality</u>:

$$\langle n_1|n_2\rangle = C_{n_1}\delta_{n_1 n_2} \quad \leftarrow \text{Dirac Notation}$$

or

$$\int_{-L/2}^{L/2} dx \left[e^{+ik_{n_1}x}\right]^* e^{ik_{n_2}x} = L\,\delta_{n_1 n_2} \quad \leftarrow \text{Conventional notation}$$

② <u>Expansion and Expansion coefficients</u>:

$$|F\rangle = \sum_n F_n \frac{|n\rangle}{C_n} \quad \text{where} \quad F_n = \langle n|F\rangle$$

Or

$$F(x) = \frac{1}{L}\sum_n F_n e^{ik_n x} \quad \text{where}$$

$$F_n = \int_{-L/2}^{L/2} [e^{ik_n x}]^* F(x)\, dx$$

③ <u>Completeness</u>

$$|F\rangle = \sum_n \frac{|n\rangle}{C_n} F_n$$

$$= \sum_n \frac{|n\rangle\langle n|F\rangle}{C_n}$$

$$= \underbrace{\left(\sum_n \frac{|n\rangle\langle n|}{C_n}\right)}_{= \mathbb{1}} |F\rangle$$

So the underlined term must be the identity operator. In the fourier case we have for x, x' in $-\frac{L}{2} \le x, x' \le L/2$

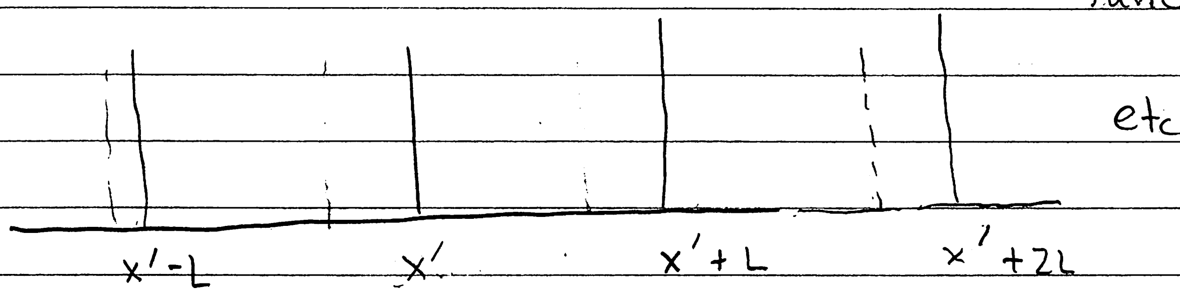
$$\frac{1}{L}\sum_n [e^{ik_n x}]^* [e^{ik_n x'}] = \delta(x-x')$$

More generally we have when x, x' not in $-\frac{L}{2} \le x, x' \le \frac{L}{2}$ we have

$$\boxed{\frac{1}{L} \sum_n e^{-ik(x-x')} = \sum_m \delta(x - x' + mL)}$$

This is the identity in the space of periodic functions

Picture:



The Fourier Transform from Fourier Series

Taking $L \to \infty$ we can derive the the fourier transform. The sum becomes an integral:

$$\frac{1}{L} \sum_n \xrightarrow[L\to\infty]{} \frac{1}{L} \int dn = \frac{1}{L} \int_{-\infty}^{\infty} \frac{L\,dk}{2\pi} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}$$

So

$$F(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F(k)$$