Energy in Electrostatics, Jackson 1.11

\[ q_1 \cdot q_2 \cdot q_3 \cdot q_4 \]

The energy of a collection of charges \( \{q_1, q_2, q_3, q_4\} \):

\[ W = \frac{1}{2} \sum_i \sum_j \frac{q_i q_j}{4\pi |\vec{x}_i - \vec{x}_j|} \]

The factor of \( \frac{1}{2} \) is included because we sum over \( i \) and \( j \) rather than pairs of particles. \( W_E \) is the energy required to assemble the collection of charges.

In continuous form, the charge density is \( \rho(\vec{x}) \) and

\[ W_E = \frac{1}{2} \int \int \frac{\rho(\vec{x}) \rho(\vec{x_0})}{4\pi |\vec{x} - \vec{x_0}|} \, d\vec{x} \cdot d\vec{x_0} \]

The potential is due to Coulomb law,

\[ \varphi(\vec{x}) = \int_{\vec{x}_0} \frac{\rho(\vec{x}_0)}{4\pi |\vec{x} - \vec{x}_0|} \]

So

\[ W_E = \frac{1}{2} \int \rho(\vec{x}) \varphi(\vec{x}) \]
Now use the Poisson equation

\[- \nabla^2 \varphi(x) = \rho\]

Then

\[W_E = \frac{1}{2} \int \left[ - \nabla \cdot (\varphi \nabla \varphi) \right] \varphi \, dx\]

Integrating by parts (see below for further detail)

\[W_E = \frac{1}{2} \int \left[ - \nabla \cdot (\varphi \nabla \varphi) \right] \left[ - \nabla \cdot (\varphi \nabla \varphi) \right] \varphi \, dx\]

\[E_i'(x) \quad E_j'(x)\]

and find

\[W_E = \frac{1}{2} \int \varphi^2 \, dx\]

Thus we see that the electrostatic energy density is

\[\varphi^2 = \frac{1}{2} \nabla \cdot (\varphi \nabla \varphi)\]

Details: we write \(- \nabla \cdot (\varphi \nabla \varphi)\) as a divergence.

Then using the divergence theorem:

\[\int_V - \varphi \cdot (\nabla \varphi) + \varphi \cdot \nabla \varphi = \int \nabla \cdot (\varphi \nabla \varphi) + \int \varphi \cdot \nabla \varphi\]
The surface integral $\to 0$ as the volume becomes large, since the fields fall sufficiently rapidly as $r \to \infty$, i.e., $E \to 0$

So:

$$\oint_{\partial V} [\Phi(x)] \cdot [\Phi(x)] = \int_{\partial V} \phi \cdot \nabla \phi$$

The general rule when integrating by parts, (and throwing away surface terms) is to move the derivative from one term to the other and flip sign.

$$(-\nabla \cdot \Phi) \Phi \quad \to \quad (\nabla \cdot \Phi)(\nabla \cdot \Phi)$$