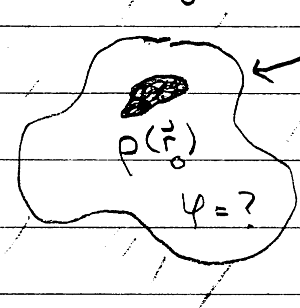
Green Functions and the Boundary Value Problem (Jackson 1.10)

- So far we considered the following case (with vanishing boundary conditions)



$\leftarrow \varphi = 0$ on boundary

The Green Function, $G_D(\vec{r}, \vec{r}_0)$ vanishes on the boundary

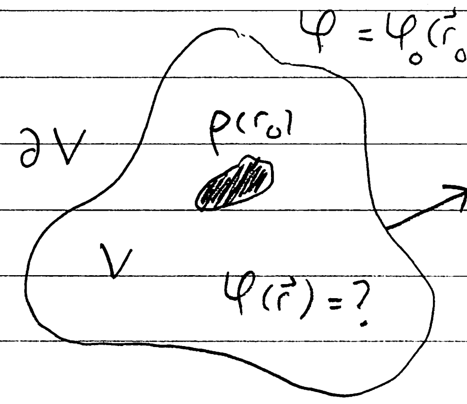
Then we said we should construct the Dirichlet green function, $G_D(\vec{r}, \vec{r}_0)$. Then for some arbitrary $\rho(\vec{r}_0)$:

$$\varphi(\vec{r}) = \int d^3 r_0 \; G_D(\vec{r}, \vec{r}_0) \, \rho(\vec{r}_0) \, .$$

$\varphi(r)\big|_{boundary} = 0$, since $G_D(\vec{r}, r_0)$ vanishes on boundary.

- More generally, we can use the Green function to construct the solution to a more general boundary problem:

$\varphi = \varphi_0(\vec{r}_0)$ is specified on the boundary



$\vec{n} \equiv$ outward directed normal

Then, take $G_D(\vec{r}, \vec{r}_0)$ to be the same Dirichlet green function as before (i.e. the boundary vanishing green function).

Then we will show that in general

$$\varphi(r) = \int_{\text{Volume}} d^3r_o \, G_D(\vec{r}, \vec{r}_o) \rho(\vec{r}_o)$$

this is the potential produce by all charges in the volume V

$$- \underbrace{\int_{\partial V} da_o \frac{\partial G_D}{\partial n_o}(\vec{r}, \vec{r}_o) \, \varphi_o(r_o)}$$

Where

$$\frac{\partial G_D(\vec{r}, \vec{r}_o)}{\partial n_o} \equiv \hat{n} \cdot \vec{\nabla}_{r_o} G(\vec{r}, \vec{r}_o)$$

$$= n_i \frac{\partial G(r, r_o)}{\partial r_o^i}$$

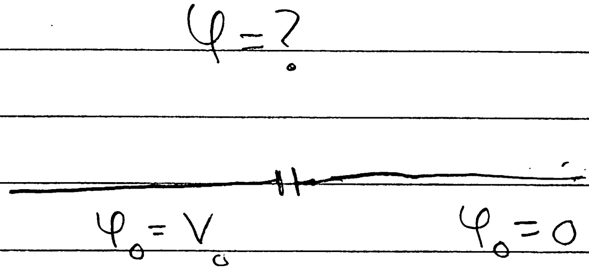
Surface integral. The boundary value, $\varphi_o(r_o)$, acts as a source for the interior. The boundary-to-bulk green function is given by the normal derivative of the Dirichlet Green Function

$$\frac{\partial G_D(\vec{r}, \vec{r}_o)}{\partial n_o} = \text{"Surface green function"}$$

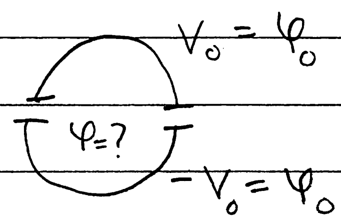
We will first use this formula in a specific example. Then we will give a proof of the theorem

Example Problems Where Green Thrm is useful

(1) Find the potential in upper half plane. Two halves of a metal plane are held at potentials $\varphi_0 = V_0$ and $\varphi_0 = 0$. See notes for solution



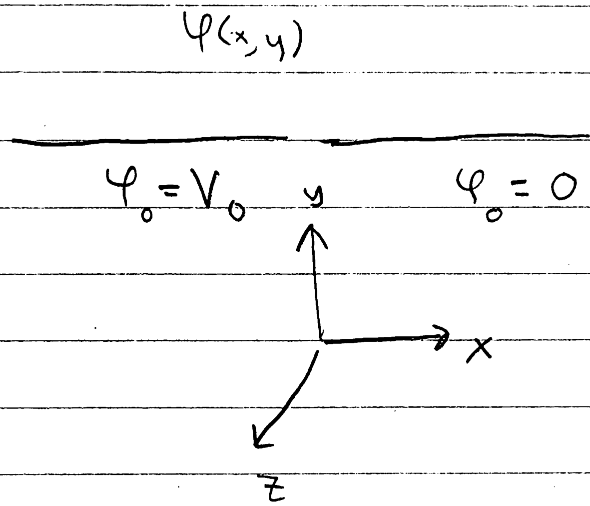
(2) Consider a cylinder (metal) infinite in length. The top half is held at potential V_0 and the bottom half is held at potential $-V_0$



Find the potential in interior. See Jackson probs 2.12 and 2.13, for solution

Example. Using the Green Theorem:

Consider the following setup. Two semi-infinite sheets are maintained at potentials $\varphi_0 = 0$ and $\varphi_0 = V_0$. Determine the potential every where, in the upper half plane



Solution:

- First some dimensional analysis. The only dimensional number in this problem is V_0. Thus $\varphi(x,y)$ must be of the form

$$\varphi(x,y) = V_0 \, f(x/y)$$

dimensionless function of the dimensionless parameter y/x

- Now use the green theorem. We first find the Dirichlet Green function, i.e. the Green function which vanishes on the boundary

$+1 \quad (x_0, y_0, z_0) = \vec{r}_0$

$\varphi = 0$

$-1 \quad (x_0, -y_0, z_0) = \vec{r}_{0I}$

$$G_D(\vec{r}, \vec{r}_0) = \left[\frac{1}{4\pi |\vec{r} - \vec{r}_0|} + \frac{-1}{4\pi |\vec{r} - \vec{r}_{0I}|} \right]$$

Then

$$\varphi(\vec{r}) = -\int_{\substack{\text{boundary}\\ \text{Surface}}} da_0 \frac{\partial G_D(\vec{r},\vec{r}_0)}{\partial n_0} \varphi_0(r_0)$$

Then we integrate over the boundary, $\underline{V_0 = \varphi_0}$ $\underline{0 = \varphi_0}$, only the left half contributes since φ_0 is zero on the right half. We have

$$\frac{\partial G(\vec{r}, r_0)}{\partial n_0} = \vec{n}\cdot\nabla_{r_0} G(\vec{r},\vec{r}_0)\Big|_{r_0 \text{ on boundary}}$$

$\vec{n} \equiv$ outward directed normal

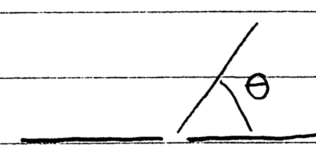
$$= \left(\frac{-\partial}{\partial y_0}\right)\left[\frac{1}{4\pi}\frac{1}{\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{1/2}} - \frac{1}{4\pi}\frac{1}{\left((x-x_0)^2+(y+y_0)^2+(z-z_0)^2\right)^{1/2}}\right]_{y_0=0}$$

and,

$$\int_{\text{boundary}} da_0 = \int_{-\infty}^{\infty} dz_0 \int_{-\infty}^{0} dx_0 = \text{integral over the left hand plate}$$

The rest is algebra (see handout). Find

$$\varphi(\vec{r}) = \frac{V_0}{\pi}\tan^{-1}\left(\frac{y}{x}\right) = \frac{V_0}{\pi}\theta$$



This satisfies the boundary condition

I. FINISHING UP PROBLEM ON GREEN THEOREM

First we have

$$\varphi(\boldsymbol{x}) = -\frac{V_o}{4\pi}\int_{-\infty}^{\infty} dz_o \int_{-\infty}^{0} dx_o \frac{-\partial}{\partial y_o}\Big[\frac{1}{((x-x_o)^2+(y-y_o)^2+(z-z_o)^2)^{1/2}} - \frac{1}{((x-x_o)^2+(y+y_o)^2+(z-z_o)^2)^{1/2}}\Big]_{y_o=0} \tag{1.1}$$

In the first step we integrate over z_o getting

$$\underbrace{\varphi(\boldsymbol{x}) = -\int_{-\infty}^{0} dx_o V_o \frac{-\partial}{\partial y_o}\left[-\frac{1}{2\pi}\log(\sqrt{(x-x_o)^2+(y-y_o)^2}) + \frac{1}{2\pi}\log(\sqrt{(x-x_o)^2+(y+y_o)^2})\right]_{y_o=0}}_{\text{Green theorem in 2D!}} \tag{1.2}$$

Now we perform do the differentiation with respect to y_o; then set $y_o = 0$, yielding

$$\varphi(\boldsymbol{x}) = \frac{V_o}{4\pi}\int_{-\infty}^{0} dx_o \frac{4y}{(x-x_o)^2+y^2} \tag{1.3}$$

Finally doing the integral over x_o we have

$$\varphi(\boldsymbol{x}) = \frac{V_0}{2\pi}\left(\pi - 2\mathrm{atan}(x/y)\right) \tag{1.4}$$

We can use some geometric identities of the arctan

$$\mathrm{atan}(x/y) = \frac{\pi}{2} - \mathrm{atan}(y/x) \tag{1.5}$$

yielding

$$\varphi(\boldsymbol{x}) = \frac{V_0}{\pi}\mathrm{atan}(y/x) \tag{1.6}$$

Remarks:

- This satisfies the boundary conditions.
- As might have been anticipated the solution is only a function of y/x. This could have been anticipated on the basis of dimensional analysis. There is no other length scale L so that the potential could be written as $\varphi(\boldsymbol{x}) = f(x/L, y/L)$. Further the only quantity which has dimensions of voltage is V_o thus from the get go we know that

  $$\varphi(\boldsymbol{x}) = V_o\, f(y/x) \tag{1.7}$$

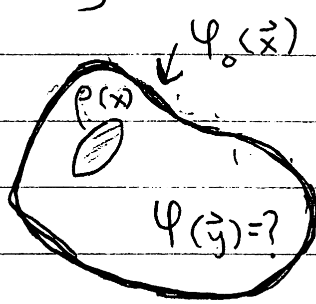
  Another way to approach this problem is just substitute this form into the Laplace equation and integrate to determine $f(y/x)$.
- Differentiating the potential to find the electric field

  $$\sigma = E_y\big|_{y=0} = -\frac{\partial}{\partial y}\varphi(\boldsymbol{x}) = \frac{-V_o}{x} \tag{1.8}$$

  This seems reasonable to me.

Proof of Green Theorem - Jackson 1.10

The proof given below generalizes to many types of equations. I used it for scalar waves in a curved geometry in my life.



$G_D(\vec{y},\vec{x})$ is the green function

$\vec{x} \equiv$ where the charge is

$\vec{y} \equiv$ where the potential is observed

Then below $\vec{\nabla}_x \equiv \frac{\partial}{\partial x^i} \equiv \partial_i$, and $\vec{\nabla} G \equiv \nabla_x G_D(y,\vec{x})$

similarly $\partial_i G \equiv \frac{\partial}{\partial x^i} G_D(\vec{y},\vec{x})$.

Now

$$-\nabla_x^2 G_D(y,x) = \delta^3(\vec{y}-\vec{x})$$

So

$$\varphi(\vec{y}) = \int d^3x \; \varphi(x) \left(-\nabla_x^2 G(y,x)\right)$$

Now integrate twice by parts so $-\nabla_x^2$ acts on $\varphi(x)$.
Using $u\,dv = d(uv) - v\,du$ twice we have:

$$\varphi(-\partial_i \partial^i G) = -\partial_i(\varphi \partial^i G) + \partial_i \varphi \partial^i G$$

$$= -\partial_i(\varphi \partial^i G) + \partial_i(\partial^i \varphi G) - (\partial_i \partial^i \varphi) G$$

$$= -\partial_i(\varphi \partial^i G - \partial^i \varphi G) + (-\nabla^2 \varphi) G$$

So using $-\nabla^2 \varphi = \rho$ we have

$$\varphi(y) = -\int_{\text{surface}} da_i \, (\varphi \, \partial^i G - \partial^i \varphi \, G) + \int_{\text{Volume}} \rho(x) \, G(y,x) \, d^3x$$

Since for the Dirichlet Green function $G_D(\vec{y}, \vec{x}) = 0$ when $\vec{x}$ or $\vec{y}$ is on the surface, this term vanishes.

Leaving

$$\boxed{\varphi(\vec{y}) = -\int_{\text{surface}} da \, \vec{n} \cdot \nabla_x \vec{G}(y,x) \, \varphi_0(x) + \int_{\text{Volume}} \rho(\vec{x}) \, G(\vec{y}, \vec{x}) \, d^3\vec{x}}$$