Solving For Green Functions with Separation

Consider Solving for the free green function in spherical coordinates

\[ G_0(\vec{r}, \vec{r}_0) = \frac{1}{4\pi |\vec{r} - \vec{r}_0|} \delta^3(\vec{r}) \]

and

\[ -\nabla^2 G(\vec{r}, \vec{r}_0) = \frac{1}{r^2} \delta(r - r_0) \delta(\cos \theta - \cos \theta_0) \delta(\phi - \phi_0) \]

1 directions // directions

Take the two // directions and write them using completeness

\[ \delta(\cos \theta - \cos \theta_0) \delta(\phi - \phi_0) = \sum_{l m} Y_{lm}^*(\theta, \phi) Y_{lm}(\theta_0, \phi_0) \]

Also expand \( G \) in the form

\[ G(\vec{r}, \vec{r}_0) = \sum_{l m} g_{lm}(r, r_0) Y_{lm}(\theta, \phi) Y_{lm}^*(\theta_0, \phi_0) \]

Then \(-\nabla^2 G = \delta^3(\vec{r})\) leads to an equation for \( g_{lm}(r, r_0) \). Using

\[ -\nabla^2 = \left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{l^2}{r^2} \right] \quad \text{and} \quad \nabla^2 Y_{lm} = \ell(\ell + 1) Y_{lm} \]
We find

\[ Y_{lm}(\theta, \phi) Y^{*}_{lm}(\theta_0, \phi_0) \left[ -\frac{1}{r^2} \frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2} \right] g_{l}(r, r_0) = \frac{1}{r_0^2} S(r - r_0) \]

\[ \begin{align*}
\Rightarrow \left[ -\frac{1}{r^2} \frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2} \right] g_{l}(r, r_0) &= \frac{1}{r_0^2} S(r - r_0) \\
\end{align*} \]

This is the 1D Green function for the problem.

We solved an identical problem on Friday:

\[ g_{l}(r, r_0) = \begin{cases} 
A r^l & r < r_0 \\
\frac{B}{r} r^{l+1} & r > r_0 
\end{cases} \]

Continuity and the jump condition

\[ (-r^2 \frac{\partial}{\partial r} g_{l}) - (-r^2 \frac{\partial}{\partial r} g_{l}) = 1 \]

(which we got by integrating across the \( \delta \)-fcn) gives enough info to solve for \( A + B \).
\[
q_{e}(r, r_0) = \begin{cases} 
\left( \frac{r}{r_0} \right)^{l+1} \frac{1}{r_0}, & r < r_0 \\
\left( \frac{r}{r_0} \right)^{l+1} \frac{1}{r_0 (2l+1)} + \frac{r}{r_0} \left( \frac{r}{r_0} \right)^{l+1} \frac{1}{r (2l+1)}, & r > r_0 
\end{cases}
\]

This can be written:

\[
q_{e}(r, r_0) = \left[ \left( \frac{r}{r_0} \right)^{l+1} \frac{1}{r_0} \right] \Theta(r_0 - r) + \left( \frac{r}{r_0} \right)^{l+1} \frac{1}{r} \Theta(r - r_0)
\]

Or equivalently as

\[
q_{e}(r, r_0) = \left( \frac{r}{r_0} \right)^{l+1} \frac{1}{r} \delta_{2l+1} \text{ same as theta fn expression}
\]

where \( r_0 \) and \( r \) are the greater and lesser of \( r \) and \( r_0 \) respectively.

Summary

\[
\frac{1}{4\pi \sqrt{r^2 - r_0^2}} \left( \frac{r_0}{r} \right)^{l+1} \frac{1}{r_0} \sum_{\ell = 0}^{\infty} \sum_{m = -\ell}^{\ell} Y_{\ell m}(\theta, \phi) Y_{\ell m}^{*}(\theta_0, \phi_0)
\]

The same technique can be used in other coordinate systems.