Solving For Green Functions with Separation

Consider Solving for the free green function in spherical coordinates

$$G_0(\vec{r}, \vec{r}_0) = \frac{1}{4\pi|\vec{r} - \vec{r}_0|}$$

and

$$-\nabla^2 G(\vec{r}, \vec{r}_0) = \overbrace{\underbrace{\frac{1}{r^2}\delta(r-r_0)}_{\perp \text{ directions}} \underbrace{\delta(\cos\theta - \cos\theta_0)\delta(\phi-\phi_0)}_{/\!/ \text{ directions}}}^{\delta^3(\vec{r})}$$

Take the two // directions and write them using completeness

$$\delta(\cos\theta - \cos\theta_0)\,\delta(\phi - \phi_0) = \sum_{lm} Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta_0,\phi_0)$$

Also expand G in the form

$$G(\vec{r}, \vec{r}_0) = \sum_{lm} g_{lm}(r, r_0)\, Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta_0, \phi_0)$$

Then $-\nabla^2 G = \delta^3(r)$ leads to an equation for $g_{lm}(r,r_0)$. Using

$$-\nabla^2 = \left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{L^2}{r^2}\right] \quad \text{and} \quad L^2 Y_{lm} = l(l+1) Y_{lm}$$

We find

$$Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta_0,\phi_0) \left[\frac{-1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{l(l+1)}{r^2} \right] g_{lm}(r,r_0)$$

$$= \frac{1}{r_0^2}\,\delta(r-r_0)\, Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta_0,\phi_0)$$

i.e.

$$\star \qquad \left[\frac{-1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{l(l+1)}{r^2} \right] g_l(r,r_0) = \frac{1}{r_0^2}\,\delta(r-r_0)$$

This is the 1D green function for the problem. We solved an identical problem on friday

$$g_l(r,r_0) = \begin{cases} A r^l & r < r_0 \\ B/r^{l+1} & r > r_0 \end{cases}$$

Continuity, and the jump condition

$$\left(-r^2 \frac{\partial}{\partial r} g_l\right) - \left(-r^2 \frac{\partial}{\partial r} g_l\right) = 1$$

(which we got by integrating across the δ-fcn) gives enough info to solve for A + B.

$$g_\ell(r, r_0) = \begin{cases} \left(\frac{r}{r_0}\right)^\ell \frac{1}{r_0} \frac{1}{(2\ell+1)}, & r < r_0 \\ \left(\frac{r_0}{r}\right)^\ell \frac{1}{r} \frac{1}{(2\ell+1)}, & r > r_0 \end{cases}$$

This can be written:

$$g_\ell(r, r_0) = \left[\left(\frac{r}{r_0}\right)^\ell \frac{1}{r_0} \Theta(r_0 - r) + \left(\frac{r_0}{r}\right)^\ell \frac{1}{r} \Theta(r - r_0)\right] \Big/ (2\ell+1)$$

Or equivalently as

$$g_\ell(r, r_0) = \left(\frac{r_<}{r_>}\right)^\ell \frac{1}{r_>} \frac{1}{2\ell+1} \longleftarrow \text{same as theta fcn expression}$$

where $r_>$ and $r_<$ are the greater and lesser of r and r_0 respectively.

Summary

$$\boxed{\frac{1}{4\pi |\vec{r} - \vec{r}_0|} = \sum_{\ell m} \frac{1}{(2\ell+1)} \left(\frac{r_<}{r_>}\right)^\ell \frac{1}{r_>} Y_{\ell m}(\theta, \phi)\, Y_{\ell m}(\theta_0, \phi_0)}$$

The same technique can be used in other coordinate systems