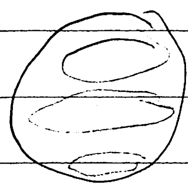
Legendre Polynomials

For azimuthally symmetric problems don't need all Y_{lm}. Since $Y_{lm} \propto e^{im\phi}$ we need only $m=0$.



$$Y_{l0} = \sqrt{\frac{2l+1}{4\pi}} \underbrace{P_l(\cos\theta)}_{\text{a polynomial in } \cos\theta}$$

Any function of θ, can be expanded in legendre polynomials (see handout):

(1) $$f(\cos\theta) = \sum_l f_l \left(\frac{2l+1}{2}\right) P_l(\cos\theta)$$ (expansion)

(2) Orthogonality (orthogonality)

$$\int_{-1}^{1} d(\cos\theta)\, P_l(\cos\theta)\, P_{l'}(\cos\theta_0) = \frac{2}{2l+1}\,\delta_{ll'}$$

(3) $$f_l = \int_{-1}^{1} d\cos\theta\, P_l(\cos\theta)\, f(\cos\theta)$$ (coeffincient)

(4) $$\sum_l P_l(\cos\theta)\, P_l(\cos\theta_0)\, \frac{2l+1}{2} = \delta(\cos\theta - \cos\theta_0)$$ (completeness)

Examples

$P_0 = 1$ $\qquad P_2 = \frac{1}{2}(3x^2-1)$

$P_1 = x$ $\qquad$ etc

Legendre Polynom

Similarly, for azimuthally symmetric systems

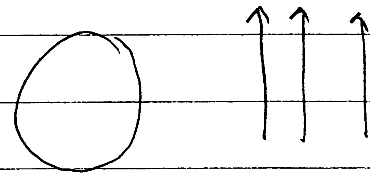
$$\varphi(\vec{r}) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

where A_l, and B_l are adjusted to match the boundary conditions

(The normalization coefficient $\sqrt{(2l+1)/4\pi}$ has been absorbed into the A's and B's)

In class problem pg 1

- A neutral metal sphere of radius a is placed in an electric field



$$E = E_0 \hat{z}$$

Determine the potential Everywhere

Solution

- Boundary Conditions $\varphi = \text{const}$ on surface, and $\varphi \xrightarrow[r \to \infty]{} -E_0 z = -E_0 r\cos\theta + \varphi_0$

could set to zero

- Outside:

$$\varphi^{out}(r) = \sum_\ell \left(A_\ell^{out} r^\ell + \frac{B_\ell^{out}}{r^{\ell+1}}\right) P_\ell(\theta)$$

$r \to \infty$ Boundary conditions: $A_\ell^{out} = 0$, Except A_1^{out}

$A_1^{out} = -E_0 r\cos\theta$, and in principle $A_0^{out} = \varphi_0$

$$\varphi^{out} = -E_0 r\cos\theta + \sum_\ell \frac{B_\ell^{out}}{r^{\ell+1}} P_\ell(\cos\theta) + \varphi_0$$

Now since the sphere is a metal surface

$$\varphi^{out} = \text{const} \quad \text{as} \quad r \to a$$

This means that $B_\ell = 0$ unless $\ell = 1$ or $\ell = 0$

$$\varphi^{out} = \varphi_0 + -E_0 r\cos\theta + \frac{B_1}{r^2}\overbrace{P_1(\cos\theta)}^{\cos\theta}$$

$$\varphi^{out} = \varphi_0 + -E_0 r\cos\theta + \frac{B\cos\theta}{r^2}$$

Requiring that $\varphi = \text{const}$ at $r = a$, sets $B = a^3 E_0$

$$\boxed{\varphi^{out} = \varphi_0 - E_0 r\cos\theta + \frac{a^3 E_0}{r^2}\cos\theta}$$

Similarly for $r < a$

0 (regularity)

$$\varphi^{in} = \sum_{\ell m}\left(A_\ell r^\ell + \cancel{\frac{B_{\ell m}}{r^{\ell+1}}}\right)P_\ell$$

Then since $\varphi\big|_{r=a} = \text{const}$, and continuity gives

$$\varphi^{in} = \varphi_0$$

So

$$\boxed{\varphi(r,\cos\theta) = \begin{cases} \varphi_0 - E_0 r\cos\theta + \dfrac{a^3 E_0}{r^2}\cos\theta & \text{outside} \\ \\ \varphi_0 & \text{inside} \end{cases}}$$

The charge density:

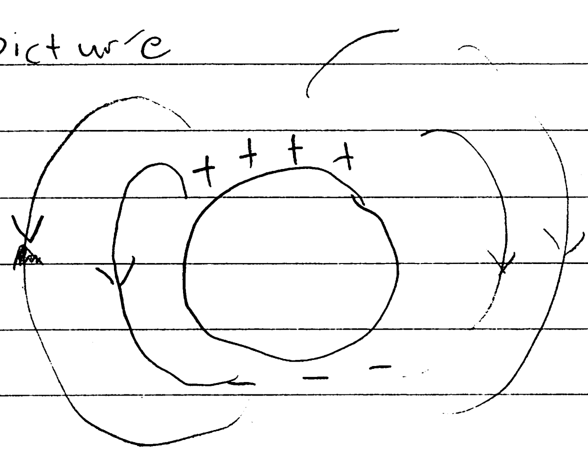
$$E_r^{out} - E_r^{in} = \sigma$$

$$-\frac{\partial \varphi^{out}}{\partial r} + \cancel{\frac{\partial \varphi^{in}}{\partial r}}^{\,0} = \sigma$$

$$\left. E_0\cos\theta + \frac{2a^3 E_0}{r^3}\cos\theta \right|_{r=a} = \sigma$$

$$\boxed{3E_0\cos\theta = \sigma}$$

So picture



To determine the dipole moment, compare the potential

$$\varphi_{ind} \approx \frac{1}{4\pi}\frac{Q_{TOT}}{r} + \frac{\vec{p}\cdot\hat{r}}{4\pi r^2} + \dots$$

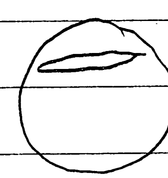
To our potential.

$$\varphi(\vec{r}) = \underbrace{\varphi_0 - E_0 r\cos\theta}_{\text{external field}} + \underbrace{\frac{a^3 E_0 \cos\theta}{r^2}}_{\text{induced field}}$$

So, $\boxed{\vec{p} = 4\pi a^3 E_0\ \hat{z}}$ by comparison of

$$\frac{\vec{p}\cdot\hat{r}}{4\pi r^2} \quad \text{and} \quad \frac{a^3 E_0 \cos\theta}{r^2}$$

One can check this by integrating over the sphere:



$$\vec{p} = \int d^3r\, \rho(\vec{r})\, \vec{r}$$

$$p_z = \int d^3r\, \rho(\vec{r})\, z$$

$$= \int \underbrace{a^2\, d\Omega}_{\text{integral over surface}}\ \sigma(\theta)\ r\cos\theta$$

$$p_z = 2\pi a^2 \int_{-1}^{1} d(\cos\theta)\,(3E_0\cos\theta)\ a\cos\theta \quad = 4\pi a^3 E_0 = p_z$$