Legendre Polynomials

For azimuthally symmetric problems don't need all $Y_{\ell m}$. Since $Y_{\ell m} \propto e^{im\phi}$ we need only $m=0$.

\[ Y_{\ell 0} = \frac{\sqrt{2\ell+1}}{\sqrt{4\pi}} P_{\ell} (\cos \theta) \]

Any function of $\theta$ can be expanded in Legendre polynomials (see handout):

1. \[ f (\cos \theta) = \sum_{\ell} f_{\ell} \left( \frac{2\ell+1}{2} \right) P_{\ell} (\cos \theta) \quad \text{(expansion)} \]

2. Orthogonality \[ \int_{-1}^{1} d(\cos \theta) P_{\ell} (\cos \theta) P_{\ell'} (\cos \theta) = \frac{2}{2\ell + 1} \delta_{\ell \ell'} \]

3. \[ f_{\ell} = \int_{-1}^{1} d\cos \theta \ P_{\ell} (\cos \theta) \ f(\cos \theta) \quad \text{(coefficient)} \]

4. \[ \sum_{\ell} P_{\ell} (\cos \theta) P_{\ell} (\cos \theta_0) \frac{2\ell+1}{2} = S (\cos \theta - \cos \theta_0) \quad \text{(completeness)} \]

Examples

<table>
<thead>
<tr>
<th>$P_0$</th>
<th>$P_2$</th>
<th>$P_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{2}(3x^2-1)$</td>
<td>$x$</td>
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</table>
Similarly, for azimuthally symmetric systems

\[ \psi(\vec{r}) = \sum_{l=0}^{\infty} (A_{\ell} r^\ell + B_{\ell}) \frac{P_{\ell}(\cos \theta)}{r^{\ell+1}} \]

where \( A_{\ell} \) and \( B_{\ell} \) are adjusted to match the boundary conditions.

(The normalization coefficient \( \frac{\sqrt{2\ell+1}}{2^\ell \Gamma(\ell+1)} \) has been absorbed into the \( A \)'s and \( B \)'s.)
In class problem pg 1

- A neutral metal sphere is placed in an electric field

\[ E = E_0 \hat{z} \]

Determine the potential everywhere

Solution

- Boundary conditions \( \Psi = \text{const} \) on surface.
and \( \Psi \to -E_0 z = -E_0 r \cos \theta + \Psi_0 \)

- Outside:

\[
\Psi^\text{out}(r) = \sum_{l} \left( A_l^\text{out} \frac{r^l}{r^{l+1}} + B_l^\text{out} \right) P_l(\cos \theta)
\]

\( r \to \infty \) Boundary conditions: \( A_0^\text{out} = 0 \), except \( A_1^\text{out} \)

\( A_1^\text{out} = -E_0 r \cos \theta \), and in principle \( A_0^\text{out} = \Psi_0 \)

\[
\Psi^\text{out} = -E_0 r \cos \theta + \sum_{l} B_l^\text{out} \frac{r^l}{r^{l+1}} P_l(\cos \theta) + \Psi_0
\]
Now since the sphere is a metal surface

\[ \Phi_{\text{out}} = \text{const as } r \to a \]

This means that \( B_\ell = 0 \) unless \( \ell = 1 \) or \( \ell = 0 \)

\[ \Phi_{\text{out}} = \Phi_0 + -E_0 r \cos \theta + \frac{B_1 P_1(\cos \theta)}{r^2} \]

\[ \Phi_{\text{out}} = \Phi_0 + -E_0 r \cos \theta + \frac{B \cos \theta}{r^2} \]

Requiring that \( \Phi = \text{const at } r = a \), sets \( B = a^3 E_0 \)

\[ \Phi_{\text{out}} = \Phi_0 - E_0 r \cos \theta + \frac{a^3 E_0 \cos \theta}{r^2} \]

Similarly for \( r < a \)

\[ \Phi_{\text{in}} = \sum_{\ell m} \left( A_\ell \frac{r^\ell}{\ell+1} + B_{\ell m} \right) P_\ell \]

Then since \( \Phi \bigg|_{r=a} = \text{const } \) and continuity gives

\[ \Phi_{\text{in}} = \Phi_0 \]
\[
\Phi(r, \cos \theta) = \begin{cases} 
\Phi_0 & \text{inside} \\
\Phi_0 - E_0 r \cos \theta + a^3 E_0 \cos \theta \over r^2 & \text{outside}
\end{cases}
\]

The charge density:

\[
E^\text{out}_r - E^\text{in}_r = \sigma
\]

\[
\frac{2 \Phi^\text{out}}{5r} + \frac{2 \Phi^\text{in}}{2 \pi} = \frac{\sigma}{r^2}
\]

\[
E_0 \cos \theta + 2a^3 E_0 \cos \theta \over r^3 = \sigma \quad \text{at} \quad r = a
\]

\[
3E_0 \cos \theta = \sigma
\]

So picture
To determine the dipole moment, compare the potential

\[ \Phi = \frac{1}{4\pi} \mathbf{Q}_{\text{tot}} + \frac{\mathbf{p} \cdot \hat{r}}{4\pi r^3} + \ldots \]

To our potential

\[ \Phi(\mathbf{r}) = \Phi_0 - E_0 r \cos \theta + \frac{a^3 E_0 \cos \theta}{r^2} \]

external field

induced field

So,

\[ \mathbf{p} = 4\pi a^3 E_0 \hat{z} \]

by comparison of

\[ \frac{\mathbf{p} \cdot \hat{r}}{4\pi r^3} \text{ and } \frac{a^3 E_0 \cos \theta}{r^2} \]

One can check this by integrating over the sphere:

\[ \mathbf{p} = \int d^3 r \rho(\mathbf{r}) \hat{r} \]

\[ \rho_z = \int d^3 r \rho(\mathbf{r}) z \]

\[ = \int a^2 \, d\Omega \, \sigma(\theta) \, r \cos \theta \]

Integral over surface

\[ \rho_z = 4\pi a^3 E_0 = \mathbf{p}_z \]