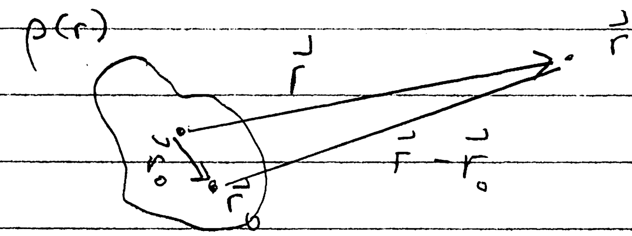
Multipole Expansion with Spherical Harmonics



Let us redo the multipole expansion

Then

$$\varphi(\vec{r}) = \int d^3\vec{r}_0 \, \frac{\rho(r_0)}{4\pi\,|\vec{r}-\vec{r}_0|}$$

For $r \gg r_0$ we can expand $r_> = r$ and $r_< = r_0$ and we have the expansion

$$\frac{1}{4\pi\,|\vec{r}-\vec{r}_0|} = \sum_{lm} \frac{r_0^{\,l}}{r^{\,l+1}} \frac{1}{(2l+1)} Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta_0,\phi_0)$$

This leads to

$$\varphi(r) = \sum_{lm} \frac{q_{lm}}{(2l+1)} \frac{Y_{lm}(\theta,\phi)}{r^{\,l+1}} = \frac{q_{00}\,Y_{00}}{r} + \frac{1}{3}\frac{q_{1m}\,Y_{1m}}{r^2} + O\left(\frac{1}{r^3}\right) \ldots$$

Where

$$q_{lm} = \int d^3 r_0 \, \rho(\vec{r}_0)\, r_0^{\,l}\, Y^*_{lm}(\theta_0,\phi_0)$$

spherical multipole moment

This multipole expansion is entirely equivalent to the expansion we had previously

$$\varphi(r) = \frac{Q_{TOT}}{4\pi r} + \frac{\vec{p}\cdot\hat{r}}{4\pi r^2} + \frac{Q_{ij}(\hat{r}^i\hat{r}^j - 1/3\,\delta^{ij})}{4\pi r^3}$$

$$+ \ldots$$

To see this one needs to understand what $Y_{lm}(\theta,\phi)$ are. Y_{lm} are linearly combos of the components of a symmetric traceless l-th rank tensor constructed out of $\hat{r}$

Cartesian	Spherical	Rank
1	Y_{00}	0
$\hat{r}_i$	Y_{1m}	1
$\hat{r}_i\hat{r}_j - \frac{1}{3}\delta^{ij}$	Y_{2m}	2
$\hat{r}^i\hat{r}^j\hat{r}^k - \frac{1}{5}(\hat{r}^i\delta^{jk} + \hat{r}^j\delta^{ki} + \hat{r}^k\delta^{ij})$	Y_{3m}	3

And so on.

To understand my meaning, take the dipole term:

$$\left.\begin{array}{l} Y_{11} \propto (\hat{x} + i\hat{y}) \\ \\ Y_{1-1} \propto (\hat{x} - i\hat{y}) \\ \\ Y_{10} \propto \hat{z} \end{array}\right\}$$

We see that Y_{1m} is a linear combo of $\hat{r}^i$

Similarly q_{1m} is a linear combo of $\vec{p}$, e.g.:

$$q_{11} = \int d^3 r_0 \; \underbrace{r_0 \, Y^*_{1m}}_{\propto (x - iy)\rho(r_0)} \rho(r_0) \propto \underbrace{p^x - i p^y}_{\text{The } x, y \text{ components of } \vec{p}}$$

since

$$p^x \equiv \int d^3 \vec{r}_0 \, x \, \rho(\vec{r}_0) \quad \text{etc.}$$

The relation between p^i and q_{1m} is the same as the relation (i.e. linear-combo) between $\hat{r}^i$ and Y^*_{1m}

The relations and normalizations are chosen so that the series agree, e.g.

$$\frac{\vec{p}\cdot\hat{r}}{4\pi r^2} = \sum_m \frac{1}{3} \frac{q_{1m}}{r^2} Y_{1m} \Rightarrow \vec{p}\cdot\hat{r} = \frac{4\pi}{3} \sum_m q_{1m} Y_{1m}$$

$2l+1$ for $l = 1$

This is the statement that

$$\vec{p}\cdot\hat{r} = \frac{(p_x - ip_y)}{\sqrt{2}}\frac{(\hat{r}_x + i\hat{r}_y)}{\sqrt{2}} + \frac{(p_x + ip_y)}{\sqrt{2}}\frac{(\hat{r}_x - i\hat{r}_y)}{\sqrt{2}} + p_z \cdot \hat{r}_z$$

$$= \frac{4\pi}{3}\left(q^*_{11} Y_{11} + q^*_{1-1} Y_{1-1} + q_{10} Y_{10}\right)$$

Similarly Y_{2m} is a linear combo of $\hat{r}^i\hat{r}^j - \frac{1}{3}\delta^{ij}$

(There are five components of $\hat{r}^i\hat{r}^j - 1/3\,\delta^{ij}$, and five $l=2$ spherical harmonics). And, q_{2m} is a linear combo of the quadrupole tensor Q_{ij} components (The map between q_{2m} and Q_{ij} is the same as between Y^*_{2m} and $\hat{r}^i\hat{r}^j - 1/3\,\delta^{ij}$). Then this map is constructed so that

$$\frac{1}{4\pi r^3} Q_{ij}\left(\hat{r}^i\hat{r}^j - \frac{1}{3}\delta^{ij}\right) = \sum_m \frac{1}{5} q_{2m} Y_{2m}/r^3$$

$\leftarrow 2l+1$ with $l=2$