Multipole Expansion with Spherical Harmonics

\[ \rho(r) \]

Let us redo the multipole expansion

Then

\[ \Psi(r) = \int d^3 r_0 \frac{\rho(r_0)}{4\pi |r - r_0|} \]

For \( r >> r_0 \) we can expand \( r_+ = r \) and \( r_- = r_0 \) and we have the expansion

\[ \frac{1}{4\pi |r - r_0|} = \sum_{lm} \frac{r_0^l}{r^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta_0, \phi_0) \]

This leads to

\[ \Psi(r) = \sum_{lm} \frac{q_{lm}}{(2l+1)} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \frac{Y_{lm}^*(\theta_0, \phi_0)}{r} + O(1/r^3) \]

where

\[ q_{lm} = \int d^3 r_0 \rho(r_0) r_0^l Y_{lm}^*(\theta_0, \phi_0) \]

spherical multipole moment
This multipole expansion is entirely equivalent to the expansion we had previously:

\[ \Psi(\mathbf{r}) = Q_{10} \mathbf{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi r^2} + \frac{Q_{20} (\mathbf{r} \cdot \mathbf{\hat{r}} - \frac{1}{3} \mathbf{r} \mathbf{\hat{r}} \cdot \mathbf{r})}{4\pi r^3} \]

+ ...

To see this one needs to understand what \( Y_{lm}(\theta, \phi) \) are. \( Y_{lm} \) are linearly combos of the components of a symmetric traceless \( \ell \)-th rank tensor constructed out of \( \mathbf{r} \).

<table>
<thead>
<tr>
<th>Cartesian</th>
<th>Spherical</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{r}_i )</td>
<td>( Y_{i0} )</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{r}_i \hat{r}<em>j - \frac{1}{3} \delta</em>{ij} )</td>
<td>( Y_{2m} )</td>
<td>2</td>
</tr>
<tr>
<td>( \hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_l )</td>
<td>( Y_{3m} )</td>
<td>3</td>
</tr>
</tbody>
</table>

And so on.
To understand my meaning, take the dipole term:

\[ Y_{11} \propto \left( \hat{x} + i \hat{y} \right) \]
\[ Y_{1,-1} \propto \left( \hat{x} - i \hat{y} \right) \]  
\[ Y_{1,0} \propto \hat{z} \]

We see that \( Y_{1m} \) is a linear combo of \( \hat{f}^i \).

Similarly, \( q_{1m} \) is a linear combo of \( \hat{p}^i \), e.g.:

\[ q_{01} = \int \frac{d^3 r_0}{r_0} \, Y_{1m}^* \rho(r_0) \propto \left( \hat{x} - i \hat{y} \right) \rho(r_0) \]

\[ \propto (x - iy) \rho(r_0) \]

The \( x, y \) components of \( \hat{p}^i \) are:

\[ \hat{p}^x = \int \frac{d^3 r_0}{r_0} \, x \rho(r_0) \quad \text{etc.} \]

The relation between \( \hat{p}^i \) and \( q_{1m} \) is the same as the relation (i.e. linear combo) between \( \hat{f}^i \) and \( Y_{1m} \).

The relations and normalizations are chosen so that the series agree, e.g.

\[ \frac{\hat{p} \cdot \hat{r}}{4\pi r^2} = \sum_m \frac{1}{3} q_{1m} Y_{1m} \Rightarrow \hat{p} \cdot \hat{r} = \frac{4\pi}{3} \sum_m q_{1m} Y_{1m} \]

\[ 2l + 1 \quad \text{for} \quad l = 1 \]
This is the statement that

\[ \vec{p} \cdot \hat{r} = (p_x - ip_y) \left( \frac{\hat{x} + i \hat{y}}{\sqrt{2}} \right) + (p_x + ip_y) \left( \frac{\hat{x} - i \hat{y}}{\sqrt{2}} \right) + p_z \cdot \hat{z} \]

\[ = \frac{4\pi}{3} \left( q_{11} Y_{11} + q_{1-1} Y_{1-1} + q_{00} Y_{10} \right) \]

Similarly, \( Y_{2m} \) is a linear combo of \( \frac{\hat{r} \cdot \hat{s}}{3} \)

(There are five components of \( \frac{\hat{r} \cdot \hat{s}}{3} \) and five \( \ell = 2 \) spherical harmonics). And, \( q_{2m} \) is a linear combo of the quadrupole tensor \( Q_{ij} \) components

(The map between \( q_{2m} \) and \( Q_{ij} \) is the same as between \( Y_{2m}^* \) and \( \frac{\hat{r} \cdot \hat{s}}{3} \)). Then, this map is constructed so that

\[ \frac{1}{4\pi r^3} Q_{ij} \left( \frac{\hat{r} \cdot \hat{s}}{3} \right) = \sum_{m} \frac{1}{5} q_{2m} Y_{2m} / r^3 \]

\[ \sim 2\ell + 1 \text{ with } \ell = 2 \]