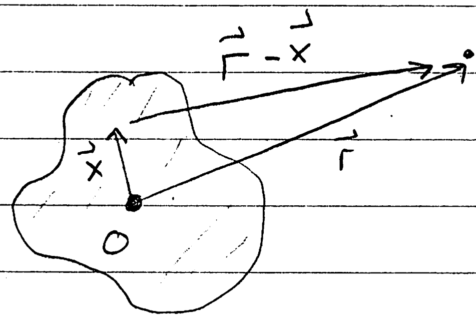
The mulipole Expansion

Given a distribution of charges, $\rho(\vec{x})$, we would like to determine the potential, $\varphi(\vec{r})$. We will determine $\varphi(\vec{r})$ for large radius. Far from the charge, $\rho(\vec{x})$, the details about the distribution don't matter and the potential is detrmined by a few moments of the distribution

A schematic is shown below. O is the origin. $\vec{x}$ labels a point on the body, $\vec{r}$ labels the observation point (i.e. where we want to know $\varphi(\vec{r})$)



The potential is

$$\star \quad \varphi(\vec{r}) = \int \frac{\rho(\vec{x})\, d^3\vec{x}}{4\pi|\vec{r}-\vec{x}|}$$

Then for $|\vec{r}| \gg |\vec{x}|$, we can expand the coulomb denominator

$$\frac{1}{|\vec{r}-\vec{x}|} = \frac{1}{(r^2 + x^2 - 2\vec{r}\cdot\vec{x})^{1/2}} = \frac{1}{r}\left(1 - 2\frac{\vec{r}\cdot\vec{x}}{r} + \frac{\vec{x}^2}{r^2}\right)^{-1/2}$$

Expanding we have for $x/r \ll 1$

$$\frac{1}{|\vec{r}-\vec{x}|} \simeq \frac{1}{r}\left[1 + \frac{\vec{r}\cdot\vec{x}}{r} + \left(\frac{3}{2}\frac{(\vec{r}\cdot\vec{x})(\vec{r}\cdot\vec{x})}{r^2} - \frac{1}{2}\frac{\vec{x}^2}{r^2}\right) + \ldots\right]$$

$$\simeq \frac{1}{r} + \frac{\hat{r}_i x^i}{r^2} + \frac{1}{2}\frac{\hat{r}_i \hat{r}_j}{r^3}(3x^i x^j - \delta^{ij}\vec{x}^2) + \ldots$$

Now take this expression and substitute into ☆, noticing that all factors of $\vec{r}$ come out of the integral

$$\varphi(r) = \frac{1}{4\pi}\Bigg[\underbrace{\frac{Q_{TOT}}{r}}_{\text{monopole}} + \underbrace{\frac{\hat{r}_i p^i}{r^2}}_{\text{dipole}} + \underbrace{\frac{1}{2}\frac{\hat{r}_i \hat{r}_j}{r^3} Q^{ij}}_{\text{quadrupole}} + \ldots\Bigg]$$

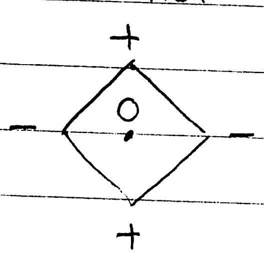
Where

$$Q_{TOT} = \int d^3x\, \rho(\vec{x}) \quad \leftarrow \text{(scalar) monopole moment}$$

$$(\vec{p})^i = \int d^3\vec{x}\, \rho(x)\, x^i \quad \leftarrow \text{(Vector) dipole moment}$$

$$Q^{ij} = \int d^3x\, \rho(\vec{x})(3x^i x^j - \delta^{ij}\vec{x}^2) \leftarrow \text{(tensor) qudrupole moment}$$

Higher terms wohld involve third rank tensors and higher. The quadrupole moment measures the anisotropy of the distribution. A sample distribution which has a net Q^{zz} is shown below



The fields are found $\vec{E} = -\vec{\nabla}\varphi(r)$

$$\vec{E}_{mono} = \frac{Q_{TOT}}{4\pi r^2}\hat{r} \propto \frac{1}{r^2}$$

$$\vec{E}_{dipole} = \frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{4\pi r^3} \propto \frac{1}{r^3}$$

$\vec{E}_{quad} \sim$ dont know $\propto \frac{1}{r^4}$