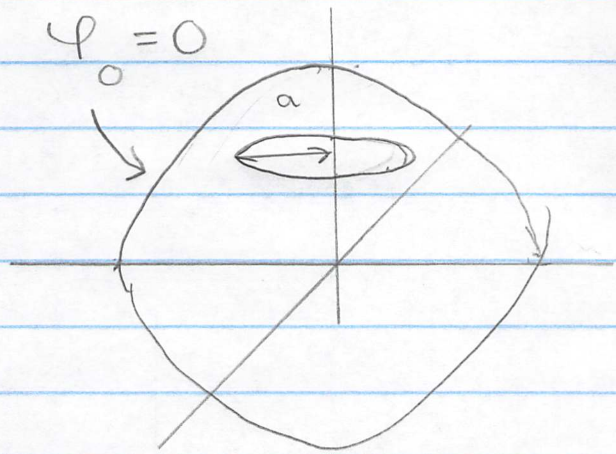
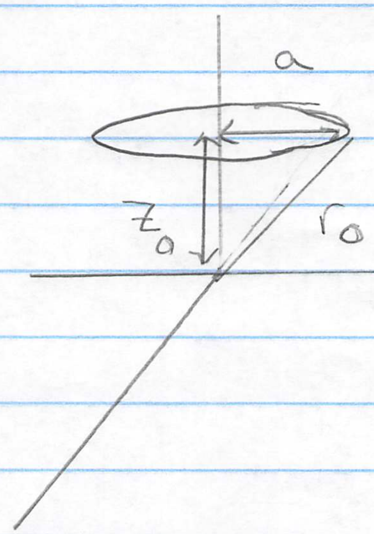
A ring in a sphere: Sturm Liouville Theory

A ring of radius, a, and charge per length λ sits inside a metal grounded sphere of radius R. Determine the force on the ring





The charge density is

$$\rho = \frac{\lambda a}{r^2} \, \delta(r - r_0) \, \delta(x - x_0)$$

where $x \equiv \cos\theta$ and $x_0 \equiv \cos\theta_0$. Then up to a constant the potential is the Green function

$$-\nabla^2 \varphi = \lambda a \, \frac{1}{r^2} \delta(r - r_0) \, \delta(\cos\theta - \cos\theta_0)$$

i.e. $\varphi = \lambda a \, G(\vec{r}, r_0)$ where

$$-\nabla^2 G(\vec{r}, r_0) = \frac{1}{r^2} \delta(r - r_0) \, \delta(\cos\theta - \cos\theta_0)$$

Mathematical Discussion - Eigenvalue Problems in 1D

· We will separate variables and all of the (second order linear equations) are of the form

weight

$$\overbrace{\left[-\frac{d}{dx}\, p(x)\, \frac{d}{dx} + q(x) \right]}^{\equiv \mathcal{L}\, y(x)} y(x) = \lambda\, w(x)\, y(x)$$

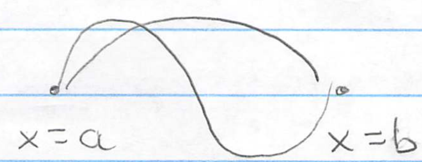
where $p(x) > 0$ and $w(x) > 0$. This is a rather general form, called Sturm Louiville form, e.g.

$$\frac{d^2 X}{dx} = -k_x^2 X$$

λ weight unity

If two homogeneous boundary conditions are specified,

e.g. $\underbrace{X(a) = X(b) = 0}_{\text{two conditions}}$



this becomes a two point eigenvalue problem like a schrödinger equation

$$\left[-\frac{d}{dx}\, p(x)\, \frac{d}{dx} + q(x) \right] \psi_n(x) = \lambda_n\, w(x)\, \psi_n$$

Due to the requirement that the wave-fcn fit in the box (two boundary conditions), only certain λ_n are allowed, e.g. $k_x = n\pi/a$ $n = 1, 2, \ldots$

Eigenvalue Problem - ring pg. 2

The eigen-fcns are complete and orthogonal

$$\int_a^b dx \underbrace{W(x)}_{\text{measure}} \psi_n(x)\, \psi_m(x) = C_n \delta_{nm}$$

And complete

$$\sum_n \frac{\psi_n(x)\, \psi_n(x_0)}{C_n} = \frac{\delta(x-x_0)}{\underbrace{W(x)}_{\text{measure}}}$$

Example of the ring in the sphere:

Separate variables

$$-\frac{r^2}{\varphi} \nabla^2 \varphi = 0 \qquad -\nabla^2 = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{-1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}$$

Try $\varphi = R(r)\,\Theta(\theta)$

$$\underbrace{\frac{-1}{R}\frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r}}_{\substack{\text{constant} \\ \text{call it } -\ell(\ell+1)}} + \underbrace{\frac{-1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta}{\partial\theta}}_{\text{constant call it } \ell(\ell+1)} = 0$$

Eigenvalue Prob pg. 3

Look at the Θ equation (Legendre Eq)

$$\left[-\frac{\partial}{\partial\theta}\underset{p(x)}{\sin\theta}\frac{\partial}{\partial\theta}\Theta\right] = \underbrace{l(l+1)}_{\lambda}\underbrace{\sin\theta}_{w(x)}\Theta \qquad (\star)$$

The regular singular points of the differential equation are when $p(x) \to 0$ (or $p(x) \to \infty$)

In this case this happens at $\theta = 0$ and $\theta = \pi$. Near the singular points there are two solutions, one regular and one irregular

$$\Theta_l(\theta) = A \underbrace{y_R^{\theta=0}}_{\substack{\text{regular}\\ \text{at } \theta=0}} + \cancelto{0}{B}\, \underbrace{y_I^{\theta=0}}_{\text{irregular}} \quad \propto \ln\theta \qquad \text{near } \theta = 0$$

Demanding regularity we set $B = 0$. Now if we integrate from $\theta = 0$ to $\theta = \pi$, at $\theta = \pi$ $y_R^{\theta=0}$ will in general be a superposition of the regular and irregular solutions at $\theta = \pi$

$$y_R^{\theta=0} \xrightarrow[\theta\to\pi]{} C\, y_R^{\theta=\pi} + D\, y_I^{\theta=\pi}$$

Eigenvalue Prob pg. 4

Only for specific values of ℓ, namely $\ell = 0, 1, 2, \ldots$, will the regular solution at $\theta = 0$ be regular at $\theta = \pi$. See handout

Thus, we have a 2-point eigenvalue problem (demanding regularity at $\theta = 0$ and regularity at $\theta = \pi$)

The eigenfunctions are:

$$\psi_n(\theta) = P_\ell(\cos\theta) \qquad \ell = 0, 1, 2, \ldots$$

Then orthogonality:

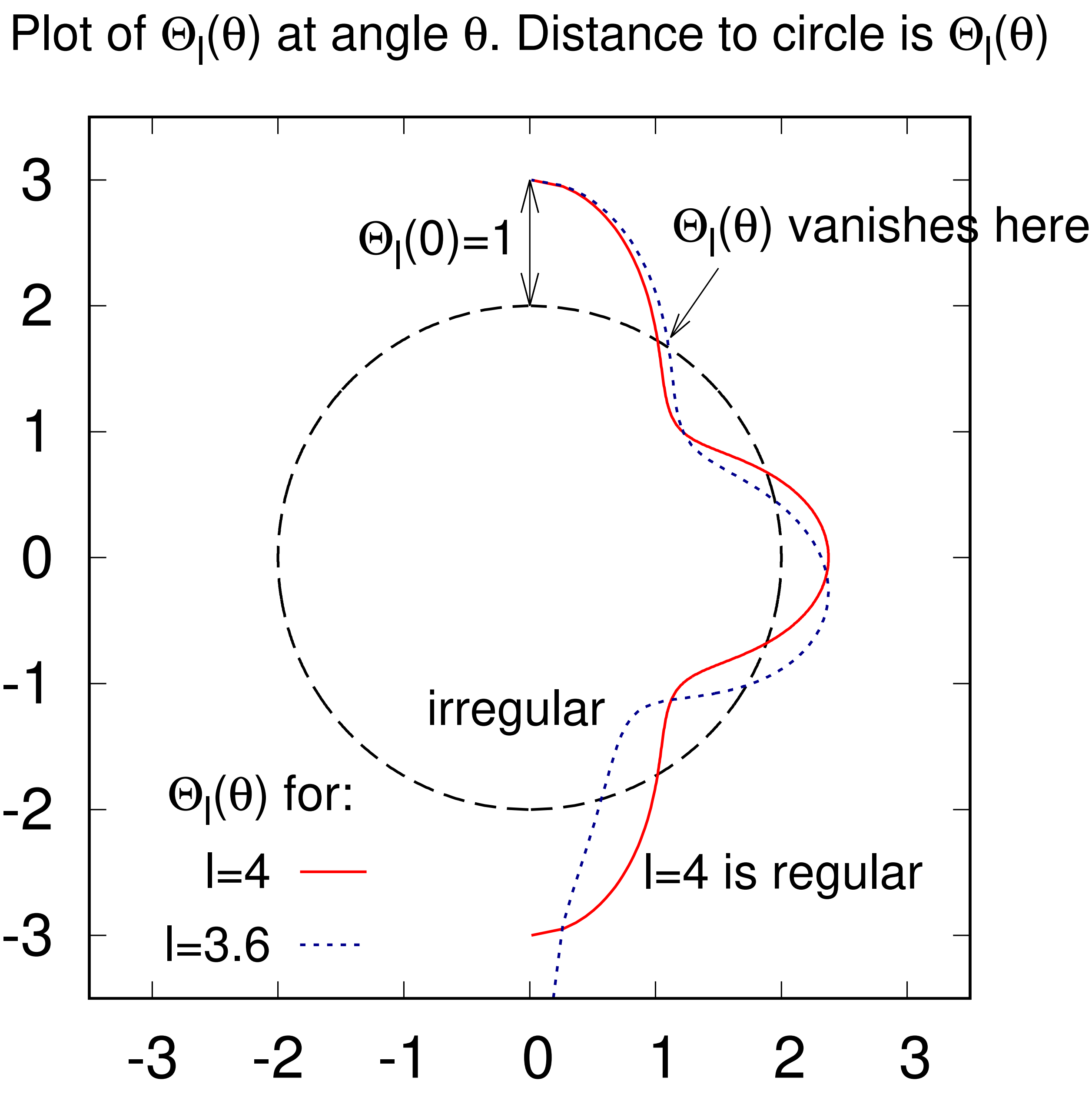
weight

$$\int_{-1}^{1} d\theta \, \sin\theta \, P_\ell(\cos\theta) \, P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1} \, \delta_{\ell\ell'}$$

and completeness:

$$\sum \frac{(2\ell+1)}{2} P_\ell(\cos\theta) \, P_\ell(\cos\theta_0) = \frac{1}{\sin\theta} \, \delta(\theta - \theta_0)$$

weight


Plot of $\Theta_l(\theta)$ at angle θ. Distance to circle is $\Theta_l(\theta)$
$\Theta_l(0)=1$
$\Theta_l(\theta)$ vanishes here
irregular
$\Theta_l(\theta)$ for:
l=4
l=3.6
l=4 is regular
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

Mathematical Discussion - 1D green functions

Returning to the ring problem we write

$$G(\vec{r},\vec{r}_0) = \sum_\ell g_\ell(r,r_0)\, P_\ell(x)\, P_\ell(x_0)\, \frac{2\ell+1}{2}$$

Then substituting this into $-\nabla^2 G = \delta^3(\vec{r}-\vec{r}_0)$ we find that g_ℓ

$$(\star\star) \qquad \left[-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{r^2} \right] g_\ell(r,r_0) = \frac{1}{r^2}\delta(r-r_0)$$

Compare to a general form

$$\star \qquad \left[-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] g(x,x_0) = \delta(x-x_0)$$

① Now note: Given two solutions to the homogeneous equations (no δ-fcn), y_{in} and y_{out}, the wronskian times $p(x)$ is constant

$$p(x)\, W(x) = p(x)\left[y_{out}\, y_{in}' - y_{in}\, y_{out}' \right] = \text{const}$$

$$= \text{independent of } x$$

Proof is easy.

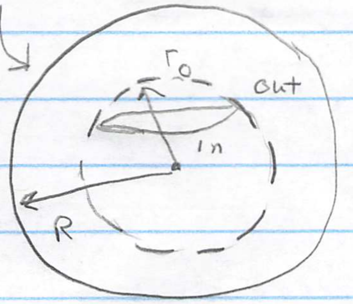
1D green - ring in sphere pg. 2

For example, for Eq ☆☆, the two homogeneous solutions are, r^{ℓ} and $1/r^{\ell+1}$

The general solution is

$$y(r) = A r^{\ell} + \frac{B}{r^{\ell+1}}$$

$\varphi = 0$ on bndry



picture of ring at r_0 in grounded sphere

Take $y_{in}(r) = \left(\frac{r}{R}\right)^{\ell}$, inside

Outside take

$$y_{out}(r) = -\left(\frac{r}{R}\right)^{\ell} + \left(\frac{R}{r}\right)^{\ell+1}$$

this vanishes on the surface

Then

$$p(r)\,W(r) = r^2\left[\, y_{out}\, y_{in}' - y_{in}\, y_{out}' \,\right]$$

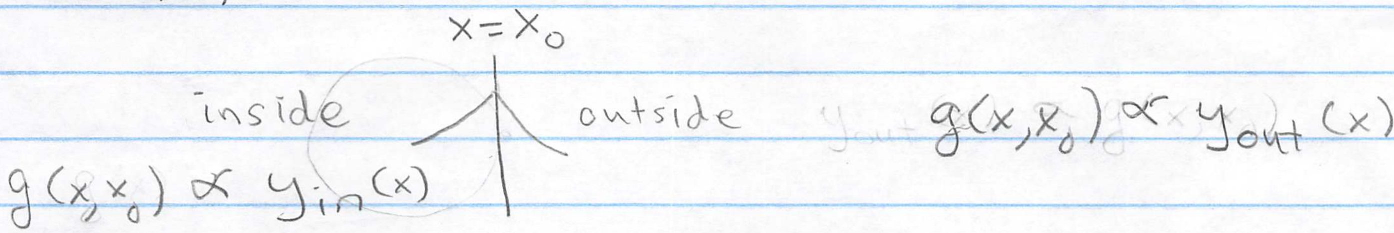
$$= +R\,(2\ell+1) \leftarrow \text{independent of } r$$

1D green pg. 3 - ring

(2) The green fcn is continuous at $x = x_0$

$$(\star\star) \qquad g(x,x_0) = C\left[\, y_{out}(x)\, y_{in}(x_0)\, \Theta(x-x_0) + y_{in}(x)\, y_{out}(x_0)\, \Theta(x_0 - x) \,\right]$$

Picture:



This satisfies the EOM inside and outside and is continuous at $x = x_0$. The constant C is adjusted to satisfy the jump condition. Integrating across the δ-fcn from Eq $\star$ two pages back

$$(\star^3) \qquad -p(x) \left.\frac{dg}{dx}\right|_{x = x_0+\varepsilon} + p(x)\left.\frac{dg}{dx}\right|_{x = x_0 - \varepsilon} = 1$$

Then substituting Eq $(\star\star)$ into Eq. $(\star^3)$

$$C\, p(x_0) \underbrace{\left[-y_{out}'\, y_{in} + y_{in}'\, y_{out} \right]_{x_0}}_{= W(x_0)} = 1$$

i.e.

$$C = \frac{1}{p(x_0)\, W(x_0)}$$

1D green - ring in sphere pg. 4

Thus finally we arrive at a very general expression for the 1D green function

$$\boxed{g(x, x_0) = \frac{y_{out}(x_>)\, y_{in}(x_<)}{p(x_0)\, W(x_0)}}$$

denominator constant!

For the problem at hand we have

$$g_\ell(r, r_0) = \underbrace{\frac{1}{R(2\ell+1)}}_{1/pW} \underbrace{\left[\left(\frac{R}{r_>}\right)^{\ell+1} - \left(\frac{r_>}{R}\right)^{\ell}\right]}_{y_{out}(r_>)} \underbrace{\left(\frac{r_<}{R}\right)^{\ell}}_{y_{in}(r_<)}$$

Leading to

$$\varphi = \sum_\ell \lambda a\, g_\ell(r, r_0)\, P_\ell(x)\, P_\ell(x_0)\, \frac{2\ell+1}{2}$$

- Use this to evalute force, energy, etc