A ring in a sphere: Sturm-Liouville Theory

A ring of radius \( a \), and charge per length \( \lambda \) sits inside a metal grounded sphere of radius \( R \). Determine the force on the ring.

The charge density is

\[
p = \lambda a \frac{S(r - r_0) S(x - x_0)}{r^2}
\]

where \( x = \cos \theta \) and \( x_0 = \cos \theta_0 \). Then up to a constant the potential is the Green function

\[
-\nabla^2 \phi = \lambda a \frac{1}{r^2} \frac{S(r-r_0) S(\cos \theta - \cos \theta_0)}
\]

i.e.

\[
\phi = \lambda a \ G(\vec{r},r_0)
\]

where

\[
-\nabla^2 G(\vec{r},r_0) = \frac{1}{r^2} \frac{S(r-r_0) S(\cos \theta - \cos \theta_0)}
\]
Mathematical Discussion - Eigenvalue Problems in 1D

We will separate variables and all of the (second order linear equations) are of the form

\[
\left[ -\frac{d}{dx} \frac{d}{dx} + q(x) \right] y(x) = \lambda \, w(x) \, y(x)
\]

where \( p(x) > 0 \) and \( w(x) > 0 \). This is a rather general form, called Sturm-Liouville form, e.g.

\[
\frac{d^2X}{dx^2} = -k_x^2 \, X
\]

If two boundary conditions are specified, e.g. \( X(a) = X(b) = 0 \),

\( x = a \) \quad \( x = b \)

this becomes a two point eigenvalue problem like a Schrödinger equation

\[
\left[ -\frac{d}{dx} \frac{d}{dx} + q(x) \right] \psi_n(x) = \lambda_n \, w(x) \, \psi_n
\]

Due to the requirement that the wave-fcn fit in the box (two boundary conditions) only certain \( \lambda_n \) are allowed, e.g. \( k_x = n \pi / a \), \( n = 1, 2, \ldots \)
The eigenfunctions are complete and orthogonal
\[
\int_a^b \psi_n(x) \psi_m(x) W(x) \, dx = \delta_{nm}\]
And complete
\[
\sum_n \psi_n(x) \psi_n(x_0) = \frac{\delta(x-x_0)}{W(x)} \quad \text{measure}
\]

Example of the ring in the sphere:

Separate variables
\[
-\nabla^2 \psi = 0 \quad -\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right)
\]

Try \( \psi = R(r) \Theta(\theta) \)

\[
\frac{-1}{R} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) + \frac{-1}{\Theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) = 0
\]

\[
\text{constant} \quad \text{call it -} L(L+1)
\]
Look at the \( \Theta \) equation (Legendre Eq)

\[
\begin{bmatrix}
-2 & \sin\theta \\
\frac{\partial}{\partial \theta} & 2
\end{bmatrix}
\begin{bmatrix}
\Theta \\
\frac{\partial \Theta}{\partial \theta}
\end{bmatrix} = l(l+1) \sin\theta \begin{bmatrix}
\Theta \\
\frac{\partial \Theta}{\partial \theta}
\end{bmatrix}
\]

\( p(x) \)

The singular points of the differential equation are when \( p(x) \to 0 \) (or \( p(x) \to \infty \)).

In this case, this happens at \( \theta = 0 \) and \( \theta = \pi \).

Near the singular points, there are two solutions, one regular and one irregular.

\[
\begin{bmatrix}
\Theta \( \theta \) \\
\frac{\partial \Theta}{\partial \theta}
\end{bmatrix} = A y_{R}^{\theta=0} + B y_{I}^{\theta=0}
\]

near \( \theta = 0 \)

regular \( \rightarrow \) irregular \( \times \ln\theta \)

at \( \theta = 0 \)

Demanding regularity, we set \( B = 0 \). Now, if we integrate from \( \theta = 0 \) to \( \theta = \pi \), at \( \theta = \pi \), \( y_{R}^{\theta=0} \) will in general be a superposition of the regular and irregular solutions at \( \theta = \pi \)

\[
y_{R}^{\theta=0} \rightarrow C y_{R}^{\theta=\pi} + D y_{I}^{\theta=\pi}
\]
Only for specific values of \( \ell \) namely \( \ell = 0, 1, 2, \ldots \) will the regular solution at \( \theta = 0 \) be regular at \( \theta = \pi \). See handout.

Thus, we have a 2-point eigenvalue problem (demanding regularity at \( \theta = 0 \) and regularity at \( \theta = \pi \)).

The eigenfunctions are:

\[
\Psi_n(\theta) = P_\ell(\cos \theta) \quad \ell = 0, 1, 2, \ldots 
\]

Then for orthogonality:

\[
\int_{-1}^{1} d\theta \sin \theta P_\ell(\cos \theta) P_{\ell'}(\cos \theta) = \frac{2}{2\ell+1} \delta_{\ell \ell'}
\]

and completeness:

\[
\sum \frac{(2\ell+1)}{2} P_\ell(\cos \theta) P_{\ell}(\cos \theta_0) = \frac{\pi}{\sin \theta} S(\theta - \theta_0)
\]
Plot of $\Theta_l(\theta)$ at angle $\theta$. Distance to circle is $\Theta_l(\theta)$

$\Theta_l(0) = 1$

$\Theta_l(\theta)$ vanishes here

irregular

$\Theta_l(\theta)$ for:

- $l=4$ is regular
- $l=3.6$
Mathematical Discussion - 1D green functions

Returning to the ring problem we write

\[ G(\mathbf{r}, \mathbf{r}_0) = \sum_l g_l(r, r_0) P_l(x) P_l(x_0) \frac{2l+1}{2} \]

Then substituting this into \(-\nabla^2 G = \delta^3(\mathbf{r} - \mathbf{r}_0)\)
we find that \(g_l\)

\[ (\star\star) \left[ -\frac{1}{r^2} \frac{d}{dr} \frac{2}{r^2} + \ell(\ell+1) \right] g_{\ell}(r, r_0) = \frac{1}{r^2} \delta(r-r_0) \]

Compare to a general form

\[ \left[ -\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] g(x, x_0) = \delta(x-x_0) \]

1) Now note: Given two solutions to the homogeneous equations (no \(S\)-fcn), \(y_{in}\) and \(y_{out}\) the Wronskian times \(p(x)\) is constant

\[ p(x) = W(x) = p(x) \left[ y_{out} y_{in}' - y_{in} y_{out}' \right] = \text{const} \]

\[ = \text{independent of } x \]

Proof is easy.
For example, for Eq. the two homogeneous solutions are $r^l$ and $1/r^{l+1}$.

The general solution is

$$y(r) = A r^l + B r^{l+1}$$

$\Psi = 0$ on boundary

Take $y_{\text{in}}(r) = \left(\frac{r}{R}\right)^l$ inside

outside take

$$y_{\text{out}}(r) = -\left(\frac{r}{R}\right)^l + \left(\frac{R}{r}\right)^{l+1}$$

picture of ring at $r_0$ in grounded sphere

This vanishes on the surface

Then

$$p(r) W(r) = r^2 \left[ y_{\text{out}} y_{\text{in}} - y_{\text{in}}' y_{\text{out}}' \right]$$

$$= + R (2l+1) \text{ independent of } r$$
2) The green fan is continuous at $x = x_0$.

\[ g(x, x_0) = C \left[ y_{\text{out}}(x) y_{\text{in}}(x_0) \Theta(x - x_0) + y_{\text{in}}(x) y_{\text{out}}(x_0) \Theta(x_0 - x) \right] \]

This satisfies the EOM inside and outside and is continuous at $x = x_0$. The constant $C$ is adjusted to satisfy the jump condition.

Integrating across the $S$-fan from Eq. (**) two pages back:

\[ -p(x) \frac{dg}{dx} \bigg|_{x = x_0 + \epsilon} + p(x) \frac{dg}{dx} \bigg|_{x = x_0 - \epsilon} = 1 \]

Then substituting Eq (**) into Eq. (***)

\[ C p(x_0) \left[ -y'_{\text{out}} y_{\text{in}}(x_0) + y_{\text{in}} y'_{\text{out}}(x_0) \right] = 1 \]

\[ = W(x_0) \]

i.e.,

\[ C = \frac{1}{p(x_0) W(x_0)} \]
Thus finally we arrive at a very general expression for the 1D green function

\[ g(x, x_0) = \frac{Y_{\text{out}}(x)}{p(x_0)} \frac{Y_{\text{in}}(x_0)}{W(x_0)} \]

denominator constant!

For the problem at hand we have

\[ g_x(r, r_0) = \frac{1}{R(2\ell+1)} \left[ \left( \frac{R}{r} \right)^{\ell+1} - \left( \frac{r}{R} \right)^{\ell} \right] \left( \frac{r}{R} \right)^{\ell} \]

Leading to \( 1/pW \) \( Y_{\text{out}}(r) \), \( Y_{\text{in}}(r) \)

\[ Q = \sum_{\ell} |a| g_x(r, r_0) P_{\ell}(x) P_{\ell}(x_0) \frac{2\ell+1}{2} \]

* Use this to evaluate force, energy, etc