Separation of Variables in Spherical Coordinates
(Jackson 3.1 - 3.3)

Now we will follow the same procedure in spherical coordinates. The Laplacian operator reads

$$-\nabla^2 = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{L^2}{r^2}$$

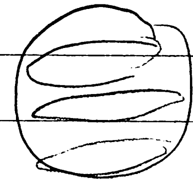
where L^2 is (without the $\hbar^2$) the squared angular momentum operator in Quantum Mechanics

$$L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{-1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Thus, in spherical coordinates the separation of variables will involve the eigen-function $Y_{\ell m}(\theta,\phi)$ (spherical harmonics) of the L^2 operator.

$$\boxed{\begin{aligned} L^2 Y_{\ell m} &= \ell(\ell+1) Y_{\ell m} \\ L_z Y_{\ell m} &= m Y_{\ell m} \end{aligned}} \qquad L_z = -i\frac{\partial}{\partial\phi}$$

The boundary conditions are specified on the θ, ϕ surface. These constitute the // directions, while the radial directions constitute the perpendicular directions



We will show in the next example that a general solution to the Laplace equation is

$$\boxed{\varphi(r,\theta,\phi) = \sum_{lm} \underbrace{\left(A_{lm}\, r^{l} + \frac{B_{lm}}{r^{l+1}}\right)}_{\text{perpendicular to surface}} \underbrace{Y_{lm}(\theta,\phi)}_{\parallel \text{ to surface}}}$$

Here

$$\sum_{lm} \equiv \sum_{l=0}^{\infty} \sum_{m=-l}^{l}$$

and the coefficients A_{lm} + B_{lm}, are adjusted to match the boundary conditions.

Let us recall the Properties of $Y_{lm}(\theta,\phi)$

① Orthogonality:

$$\int d\Omega \; Y^{*}_{lm}(\theta,\phi)\, Y_{l'm'}(\theta,\phi) = \delta_{ll'}\,\delta_{mm'}$$

where $d\Omega$ is the integral over the sphere

$$\int d\Omega \equiv \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \equiv \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi$$

② Expansion of a function

In the next problem, the charge / solid angle on the surface of a sphere is:

$$\underline{S(\theta,\phi) = \sum_{lm} S_{lm} Y_{lm}(\theta,\phi)}$$

where we have expandes $S(\theta,\phi)$ in spherical harmonics

$$\underline{S_{lm} = \int d\Omega \, Y^*_{lm} \, S(\theta,\phi)}$$

Here $S(\theta,\phi)$ is defined so that the charge

$$Q = \int d\Omega \, S(\theta,\phi)$$

③ Completeness. As with all eigenfunction we have the completeness relation $\sum_n |n\rangle\langle n| = \mathbb{1}$

$$\sum_{lm} Y^*_{lm}(\theta,\phi) \, Y_{lm}(\theta_0,\phi_0) = \underbrace{\delta(\cos\theta - \cos\theta_0)\,\delta(\phi-\phi_0)}_{\text{This is the identity on the Sphere,}}$$

e.g.

$$S(\theta,\phi) = \int d\Omega_0 \, \delta(\cos\theta - \cos\theta_0)\,\delta(\phi-\phi_0)\, S(\theta_0,\phi_0)$$

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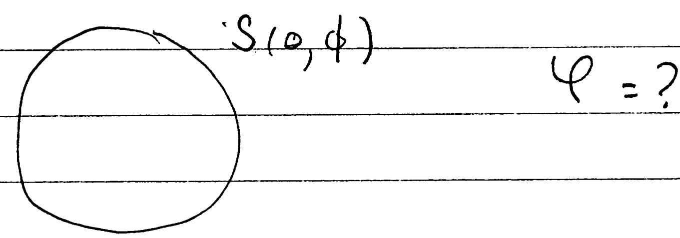
Problem

- Given a charged sphere shell of radius r_0 with charge per solid angle $S(\theta, \phi)$ determine the potential everywhere



Plan:

- Separate variables, solve inside & outside, integrate across the shell to match the inside and outside.

Solution:

Inside have, $r < R$:

$$-\nabla^2 \varphi = 0$$

Notice that if $\varphi = \underbrace{R(r)}_{\substack{\text{Coord} \\ \perp \text{ to surf}}} \underbrace{Y(\theta, \varphi)}_{\substack{\text{coords } \parallel \\ \text{to surface}}}$

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Then we compute:

$$-\frac{r^2}{\varphi}\nabla^2\varphi \quad \text{with} \quad -\nabla^2 = -\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\underbrace{L^2}_{\text{angles only}}$$

And find

$$\underbrace{-\frac{1}{R}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r}}_{\text{if } r \text{ is fixed}} + \underbrace{-\frac{1}{Y}L^2 Y}_{\text{Then this is constant}} = 0 \qquad L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Thus we are led to consider the eigenvalue eqanation

$$L^2 Y_n = \lambda_n Y_n \qquad \lambda_n = l(l+1)$$

We know this eigenvalule problem, the operator is hermitian and the eigenfcns are complete & orthogom.

Thus at each r we can expand the solution

$$\varphi(r) = \sum_{lm} R_{lm}(r)\, Y_{lm}(\theta, \varphi)$$

And adjust the $R_{lm}(r)$ to match the solution accross the shell

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Now :

for $r \to 0$ want a regular solution

$$B^{in}_{lm} = 0$$

for $r \to \infty$ want a regular solution

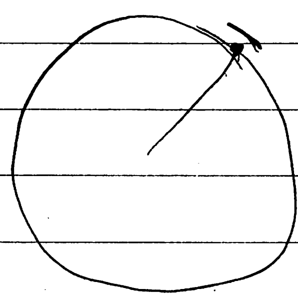
$$A^{out}_{lm} = 0$$

So for the remaining two conditions

A^{in}_{lm} B^{out}_{lm}

we demand continuity of φ, and relquire that in each surface element

$$\underbrace{\vec{n} \cdot E_{out} - \vec{n} \cdot E_{in} = \sigma}$$



this is derived by integrating the poisson equation from $R - \varepsilon$ to $R + \varepsilon$

It is simplest to use $\vec{n} \cdot (E_{out} - \vec{E}_{in}) = \sigma$ directly. But to show the procedure, we will integrate the poisson equation

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$\rho(r) = \dfrac{S(\theta,\phi)}{r^2}\delta(r-r_0)$ ← this is the Charge density charge/vol

So that

$$\int d^3\vec{r}\,\rho(\vec{r}) = \int d\Omega\, S(\theta,\phi)$$ ← $S(\theta,\phi)$ is charge/solid angle

Then the Poisson Equation is:

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}L^2\right]\sum_{lm} R_{lm}(r)Y_{lm}(\theta,\phi) = \frac{S(\theta,\phi)}{r^2}\delta(r-r_0)$$

So expanding $S(\theta,\phi)$ in the same basis:

$$S(\theta,\phi) = \sum_{lm} S_{lm} Y_{lm}(\theta,\varphi)$$

this a prototype eq for 1D green fcns

So

$$\boxed{\left[-\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{l(l+1)}{r^2}\right]R_{lm}(r) = \frac{S_{lm}}{r^2}\delta(r-r_0)}$$

Now multiply by r^2 and integrate from $r = r_0 - \varepsilon$ to $r = r_0 + \varepsilon$

- $$\int_{r_0-\varepsilon}^{r_0+\varepsilon} \frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R_{lm} = r^2\frac{\partial R^{out}_{lm}}{\partial r} - r^2\frac{\partial R^{in}_{lm}}{\partial r}$$

- $l(l+1)$ term gives $O(\varepsilon)$ since R_{lm} is continuous

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So find

$$\star \quad -\left(r^2 \frac{\partial R^{out}_{lm}}{\partial r} - r^2 \frac{\partial R^{in}_{lm}}{\partial r} \right)\Bigg|_{r=r_0} = S_{lm} \qquad \text{(jump condition)}$$

This is equivalent to $n\cdot E^{out} - n\cdot E^{in} = \sigma$. (Prove me!)
So with continuity and jump equation

$$\frac{B^{out}_{lm}}{r_0^{l+1}} = A^{in}_{lm}\, r^{l} \qquad \text{(continuity)}$$

$$(l+1)\frac{B^{out}_{lm}}{r_0^{l}} + l\, A^{in}_{lm}\, r_0^{l+1} = S_{lm} \qquad \text{(jump)}$$

Find

$$B_{lm} = \frac{S_{lm}}{2l+1}\, r_0^{l}$$

$$A_{lm} = \frac{S_{lm}}{2l+1}\,\frac{1}{r_0^{l+1}}$$

So This is it. It expresses φ in terms of $S(\theta,\phi)$

$$\boxed{\begin{aligned} \varphi(\vec r) &= \sum_{lm} \frac{S_{lm}}{2l+1}\left(\frac{r_0}{r}\right)^{l}\frac{1}{r}\, Y_{lm}(\theta,\phi) \qquad r>r_0 \\ \varphi(\vec r) &= \sum_{lm} \frac{S_{lm}}{2l+1}\left(\frac{r}{r_0}\right)^{l}\frac{1}{r_0}\, Y_{lm}(\theta,\phi) \qquad r<r_0 \end{aligned}}$$

This gives the potential for any source specified by

$$S(\theta,\varphi) = \sum_{lm} S_{lm} Y_{lm}(\theta,\phi)$$

For $S_{lm} = Y^*_{lm}(\theta_0,\phi_0)$, this a point charge (see overview)

$$\varphi(r) = \frac{1}{4\pi|\vec{r}-\vec{r}_0|} = \sum_{lm} \left(\frac{r_0}{r}\right)^l \frac{1}{r} Y_{lm}(\theta_0,\phi_0)\, Y_{lm}(\theta,\phi) \qquad r > r_0$$

Important Points

① Identify coords $\perp$ (i.e. r) and parallel (θ,φ) to surface where b.c. are specified

② Solve eigenvalue eqn for parallel directions these are complete & orthogonal

③ Expand solution in these eigen-fcns and solve for $\perp$ direction

general homogeneous solution

$$\boxed{\varphi = \sum_{lm}\left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}}\right) Y_{lm}(\theta,\phi)}$$

④ Adjust coefficients so boundary conditions are satisfied. Integrate across δ fcns with second order eqs to determine jump conditions.