Grad, Div, Curl, and Laplacian

CARTESIAN $\quad d\boldsymbol{\ell} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \qquad d^3r = dx\,dy\,dz$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\psi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

CYLINDRICAL $\quad d\boldsymbol{\ell} = d\rho\hat{\boldsymbol{\rho}} + \rho d\phi\hat{\boldsymbol{\phi}} + dz\hat{\mathbf{z}} \qquad d^3r = \rho\, d\rho\, d\phi\, dz$

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right]\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

SPHERICAL $\quad d\boldsymbol{\ell} = dr\hat{\mathbf{r}} + r d\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}} \qquad d^3r = r^2\sin\theta\, dr\, d\theta\, d\phi$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Figure 1: Grad, Div, Curl, Laplacian in cartesian, cylindrical, and spherical coordinates. Here ψ is a scalar function and $\mathbf{A}$ is a vector field.

Problem 1. Radiation from a pair of oscillators

Consider two non-relativistic charged particles of charge q separated by a distance 2ℓ moving in the x-y plane (see below). A stationary negative charge of magnitude $-2q$ remains at the origin neutralizing the system.

The trajectory of the first charged particle is harmonic with amplitude d, moving parallel to the y-axis and located at $x = \ell$

$$(x_1(t), y_1(t)) = (\ell,\, de^{-i\omega t}) \,. \tag{1}$$

The trajectory of the second charged particle is also harmonic but is located at $x = -\ell$.

$$(x_2(t), y_2(t)) = (-\ell,\, de^{-i\omega t}) \,. \tag{2}$$

You may assume $\ell \gg d$ and that $(\omega d)/c \ll 1$ throughout this problem.

(a) First assume that $\omega\ell/c \ll 1$ is small. Determine the time averaged power that is radiated per solid angle as measured by a detector placed at an angle θ in the z-x plane as shown below.

(b) Determine the instantaneous real electric field measured by a detector at time t and distance R along the z-axis and y-axes in the far field. (Your answer should be real and should be a vector.) Explain physically the origin of the different field strengths on the z and y axes.

Now assume that $k\ell$ is not small, so that a multipole expansion is not appropriate.

(c) Determine the Lorentz gauge potential $\boldsymbol{A}_{\rm rad}$ in the far field for a detector placed at an angle θ in the z-x plane.

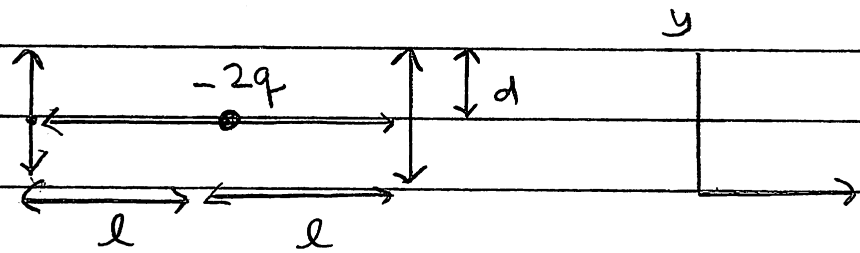
(d) Determine the power radiated per solid angle for a detector at an angle θ in the z-x plane.

(e) How would your result change if

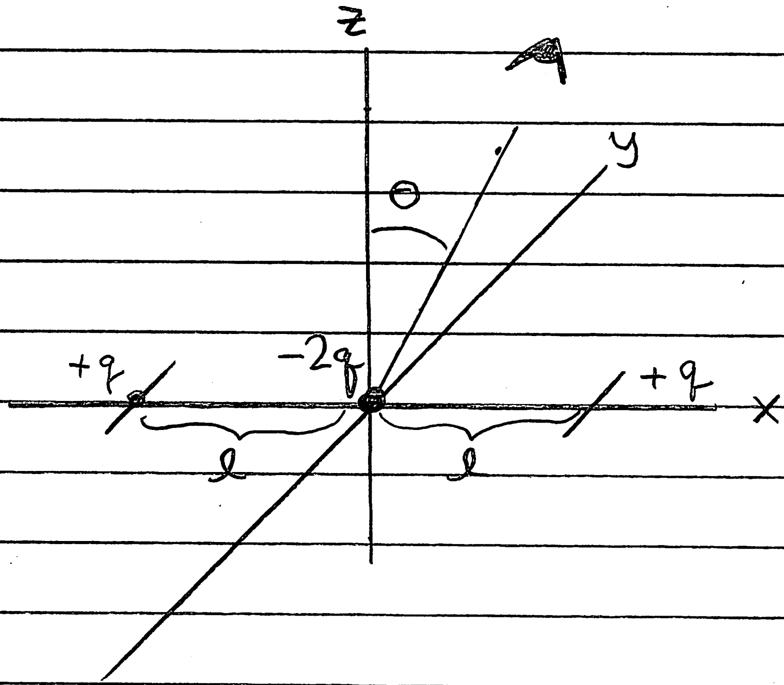
$$(x_2(t), y_2(t)) = (-\ell, -de^{-i\omega t}) \,. \tag{3}$$

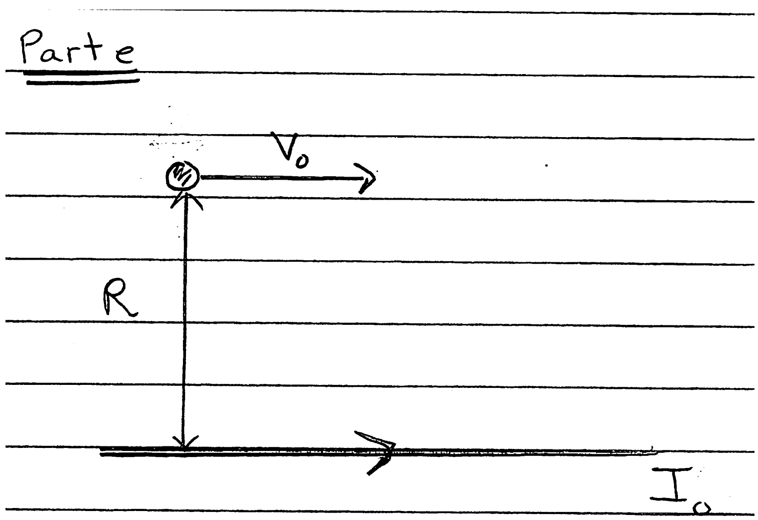
What is the leading multipole at small separation ℓ in this case, and how does the radiated power depend on frequency in this small ℓ limit.

Top View



3D -View





Problem 2. A small sphere and a wire

(a) Write down the covariant action of the electric and magnetic fields coupled to a current J^{μ}. Determine the equations of motion by varying the action. Does one obtain all of the Maxwell equations in this way? Explain.

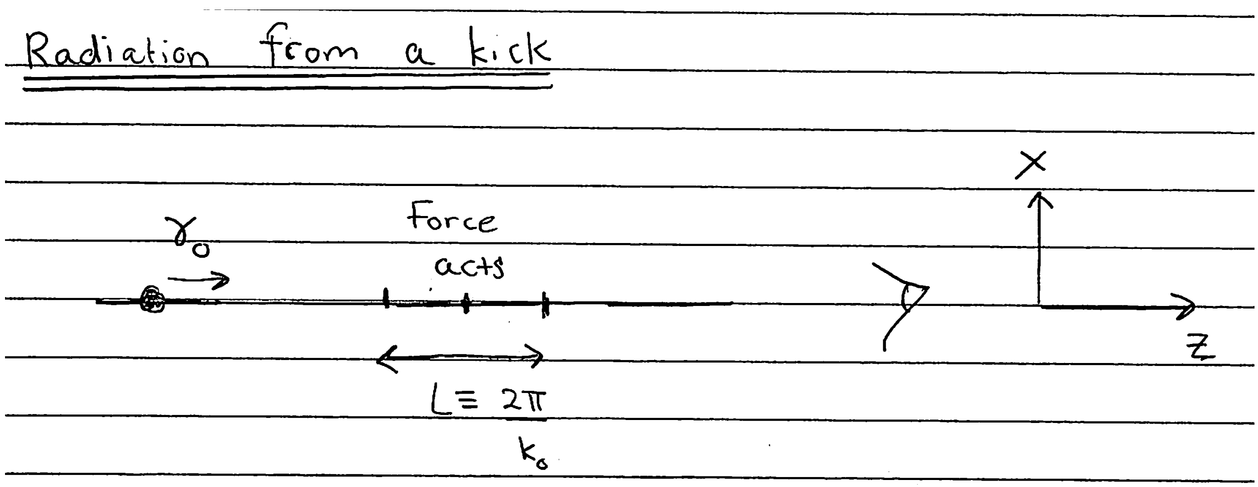
(b) What are the two Lorentz invariants quadratic in the field strength? Evaluate them in terms of $\boldsymbol{E}(t, \boldsymbol{x})$ and $\boldsymbol{B}(t, \boldsymbol{x})$.

(c) Consider a frame which has a non-zero magnetic field $\boldsymbol{B}(t, \boldsymbol{x})$, but no electric field. Using the covariant form the transformation laws of $F^{\mu\nu}$ derive the electric field $\underline{\boldsymbol{E}}(t, \boldsymbol{x})$ and $\underline{\boldsymbol{B}}(t, \boldsymbol{x})$ measured by an observer moving with velocity v along the x-axis.

(d) In part (b) you should find that $\underline{\boldsymbol{E}}$ is perpendicular to $\underline{\boldsymbol{B}}$. Explain why this must be the case.

Now consider a very small neutral metal sphere of radius a moving non-relativistically with velocity v_o parallel to a wire at radius R (see above). The wire carries a steady current I_o.

(e) Determine the force (magnitude and direction) between the sphere and the wire. (Hint: analyze the situation in the rest frame of the sphere. Express the force in terms of the induced dipole moment $\boldsymbol{p} = \alpha_E \boldsymbol{E}$ in this frame.)



Problem 3. Radiation from a harmonic kick

An ultra-relativistic relativistic charged particle (of charge q and mass m) travels in the z direction with initial energy $E_o = \gamma_o mc^2$ ($\gamma_o \gg 1$). The particle experiences a small sinusoidal force in the x direction between $-L/2$ and $L/2$:

$$F^x(z) = F_o \sin(k_o z)\,, \qquad -\frac{L}{2} < z < \frac{L}{2}\,, \tag{4}$$

where $L \equiv 2\pi/k_o$.

(a) Determine the acceleration of the ultra-relativistic particle to first order in F_o.

(b) Determine the energy per solid angle radiated in the z direction (i.e. directly forward). How does your result scale with γ_o? Work to lowest non-trivial order in F_o.

(c) Determine the total energy radiated during the process. How does your result scale with γ_o? Work to lowest non-trivial order in F_o.

(d) Determine the frequency spectrum per solid angle radiated in the z direction, i.e. determine $(2\pi)\, dW/d\omega d\Omega$ in the forward direction. Work to lowest non-trivial order in F_o.

(e) How does the typical frequency in part (d) scale with γ_o. Can you give an interpretation of this typical frequency scale? (Note: it is not necessary to do part (d) to answer this question.)