a) This is dipole radiation. The dipole moment

\[ \vec{p} = q \vec{r}_1 + q \vec{r}_2 \]

\[ \vec{p} = 2q d \, e^{i\omega t} \hat{y} \]

Then from the usual formulas \( \vec{p}(t) = \vec{p}_w \, e^{-i\omega t} \)

\[ \frac{\dd{p}}{\dd{t^2}} = \frac{1}{16\pi^2 c^3} \frac{1}{2} \left| \vec{n} \times \vec{n} \times \vec{p}_w \right|^2 \]

Now \( \vec{n} \times \vec{n} \times \vec{p}_w = -\vec{p}_w + \vec{n}(\vec{n} \cdot \vec{p}_w) \). But in this case \( \vec{n} \cdot \vec{p}_w = 0 \) since \( \vec{p}_w \propto \hat{y} \), but \( \vec{n} \) lies in the \( x-z \) plane. Thus

\[ \frac{\dd{p}}{\dd{t^2}} = \frac{1}{16\pi^2 c^3} \frac{(2qd)^2}{2} \]

b) The electric field for dipole radiation

\[ \vec{E}(t, \vec{r}) = \frac{1}{4\pi \gamma c^2} \vec{n} \times \vec{n} \times \vec{p}(t_e) \]

Now \( \vec{p}(t_e) = \vec{p}_w \cos(-\omega(t - \gamma/c)) \)

\[ \vec{p}(t_e) = -\omega^2 \vec{p}_w \cos(\omega t - kr) \]
So
\[ \vec{E}(t, \vec{r}) = \frac{\cos(\omega t - kr)}{4\pi rc^2} \hat{n} \times \hat{n} \times \hat{p}_w \left(-\omega^2\right) \]

where \( \hat{p}_w = 2\Phi q \hat{\mathbf{j}} \). On the \( z \) axis we have
\[ \hat{n} \times \hat{n} \times \hat{p}_w = \hat{\mathbf{z}} \times \hat{\mathbf{z}} \times \hat{p}_w = -\hat{p}_w \]

So
\[ \vec{E}(t, \vec{R}) = \frac{k^2 \cos(\omega t - kr)}{2\pi R} \hat{\mathbf{j}} \left(2\Phi q\right)^2 \]

On the \( y \) axis \( \hat{n} \times \hat{n} \times \hat{p}_w = 0 \)
\[ \vec{E}(t, \vec{R}) = 0 \]
(\( y \)-axis)

This is clear for \( \hat{n} \) along the \( y \)-axis the currents \( 2\Phi J \) are along the line of sight and do not drive the \( \vec{E} + \vec{B} \) fields which must be transverse to the line of sight.
c) In this case

\[ \hat{A}_{\text{rad}}(t, \vec{r}) = \frac{1}{4\pi r c} \int d^3r_0 \hat{J}(T, \vec{r}_0) \]

where

\[ \hat{J} = q \hat{v}_1(T) \hat{S}^3(\vec{r}_0 - \vec{R}_1(T)) \]

For particle 1 and particle 2

\[ \hat{v}_1(T) = -i\omega \hat{d} e^{-i\omega T} \hat{\gamma} \]

Here

\[ T = t - \vec{r} \cdot \frac{1}{c} + \hat{n} \cdot \vec{r}_0 \]

Now think physically

\[ \hat{A}_{\text{rad}}(t, \vec{r}) = \frac{\hat{\gamma}}{4\pi r c} q (-i\omega d) e^{-i\omega \left( t - \frac{\vec{r}}{c} + \frac{\vec{n} \cdot \vec{r}_0}{c} \right)} \]

\[ + \frac{\hat{\gamma}}{4\pi r c} q (-i\omega d) e^{-i\omega \left( t - \frac{\vec{r}}{c} + \frac{\vec{n} \cdot \vec{R}_1(T)}{c} \right)} \]

The positions of particle 1 and particle 2
Rad from pair pg. 4

are essentially constant in time. Just look at picture

\[ \vec{R}_1 = \vec{e} \hat{x}, \quad \vec{R}_2 = -\vec{e} \hat{x} \]

So

\[ \vec{A}_{rad} = \hat{y} \cdot q_b \cdot (-i \omega d) \cdot e^{-i \omega t + ikr} \cdot \frac{1}{4\pi rc} \cdot \left[ e^{-i \omega \frac{r}{c} \cdot \hat{n} \cdot \hat{x}} + e^{+i \omega \frac{r}{c} \cdot \hat{n} \cdot \hat{x}} \right] \] \hspace{1cm} (1)

So for \( \hat{n} = (\sin \theta, 0, \cos \theta) \)

\[ \vec{A}_{rad} = \hat{y} \cdot q_b \cdot (i \omega d) \cdot e^{+i \omega t + ikr} \cdot \frac{1}{4\pi rc} \cdot 2 \cdot \cos (\sin \theta kl) \] \hspace{1cm} (2)

So

\[ \frac{dP}{d\Omega} = \frac{c}{2} \cdot \frac{-i \omega \cdot \hat{n} \times \hat{n} \times \vec{A}_{w,rad}}{c} \]

\[ = \frac{1}{32 \pi^2 c^3} \cdot \frac{\omega^4 \cdot (2qd)^2 \cdot [\cos (\sin \theta kl)]^2}{32 \pi^2 c^3} . \]
So we note for $kl << 1$ we recover the result of part (a):

$$\frac{dP}{d\Omega} \rightarrow \frac{w^4}{32\pi^2c^3}(2qd)^2$$

e) The analysis is the same but the sign of the second term is reversed leading to (see pg. 3 Eq (★))

$$\frac{dP}{d\Omega} = \frac{w^4}{32\pi^2c^3}(2qd)^2 [\sin (\sin \theta kl)]^2$$

For $kl << 1$ we see that since $k = \frac{w}{c}$

$$\frac{dP}{d\Omega} \propto \left(\frac{w}{c}\right)^6$$

This is characteristic of quadrupole radiation.
\[ S = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{c} J^\mu A_\mu \right] \]

Varying the action \( A_\mu \rightarrow A_\mu + \delta A_\mu \)

\[ \delta S = \int d^4x \left[ -\frac{1}{2} F^{\mu\nu} \left( \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu \right) + \frac{i}{c} J^\mu \delta A_\mu \right] \]

Integrate by parts

\[ \delta S = \int d^4x \left[ +\frac{1}{2} \partial_\mu F^{\mu\nu} \delta A_\nu - \frac{1}{2} \partial_\nu F^{\mu\nu} \delta A_\mu \right] \]

\[ + \frac{i}{c} J^\mu \delta A_\mu \]

After relabelling and using \( F^{\mu\nu} = -F^{\nu\mu} \) we have

\[ \delta S = \int d^4x \left[ \partial_\nu F^{\nu\rho} + \frac{i}{c} J^\rho \right] \]

So the EOM is

\[ -\partial_\nu F^{\nu\rho} = \frac{J^\rho}{c} \]

This is not all of the Maxwell equations
The residual equations are \( \partial_{\mu} F_{\nu \rho} = 0 \) or
\[
\partial_{\mu} F_{\nu \rho} + \partial_{\nu} F_{\rho \mu} + \partial_{\rho} F_{\mu \nu} = 0
\]
the Bianchi identity. This guarantees that \( F_{\mu \nu} \) can be written
\[
F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}
\]

b) The two invariants are
\[
F_{\mu \nu} F^{\mu \nu} = -2 (E^2 - B^2)
\]
\[
F_{\mu \nu} F^{\mu \nu} = -4 E \cdot B
\]

c) The covariant transformation rule is
\[
F'^{\mu \nu} = L^\rho_{\mu} L^\sigma_{\nu} F^{\rho \sigma}
\]
Here
\[
L^\mu_{\nu} = \begin{pmatrix}
\gamma & -\gamma \beta \\
-\delta \beta & \gamma
\end{pmatrix}
\]
Sphere 3

Now

\[ F^{01} = L^0 L^1 F^{01} \]
\[ F^{01} = L^0 L^1 F^{01} = 0 \]
\[ E^x = 0 \]

\[ F^{02} = L^0 L^2 F^{12} \]
\[ E^y = -\gamma \beta \left( B^y \right) \]

and

\[ F^{03} = L^0 L^3 F^{13} \]
\[ E^z = -\gamma \beta \left( -B^y \right) \]
\[ E^z = \gamma \beta B^y \]

\[ F^{12} = L^1 L^2 F^{12} \]
\[ = L^1 L^2 F^{12} \]
\[ = L^1 L^2 F^{12} \]
\[ = \gamma F^{12} \]
\[ B^z = \gamma B^z \]
Then
\[ F_{13} = L_1^1 L_3^3 F^{\rho \sigma} \]
\[ B_y = \gamma B_y \]

And
\[ F^{23} = L_2^2 L_3^3 F^{\rho \sigma} \]
\[ F^{23} = F^{23} \quad B_x = B_x \]

To summarize
\[ E = \gamma \beta \times B \]
\[ B_{\parallel} = B_{\parallel} \]
\[ B_{\perp} = \gamma \beta \perp \]

Note
\[ E = \beta \times B \]

1) From the fact that
\[ E = \beta \times B \]
\[ E \cdot B = 0. \quad This \quad follows \quad from \quad the \quad invariant \]
Sphere 5

If

\[ F_{\mu \nu} F^{\mu \nu} = -4 E \cdot B \]

vanishes in one frame it vanishes in all. In the original frame \( E = 0 \) so \( E \cdot B \) is zero. This is true in one frame it is true in all.

e) In this case

\[ \vec{v}_0 \]

\[ \vec{z} \]

\[ \vec{x} \]

\[ \vec{I}_0 \]

The magnetic field is in the \( z \)-direction (see coordinates above)

\[ \vec{B} = \frac{I}{2\pi y} \hat{z} = B_z(y) \hat{z} \]

The \( E \)-field is with boost vector \( \vec{\beta} = + \frac{\vec{v}_0}{c} \)

\[ \vec{E} = + \frac{v_0}{c} \hat{x} \times B_z(y) \hat{z} \]

\[ \vec{E} = - \frac{v_0}{c} B_z(y) \hat{y} \]
Sphere 6

So the force

\[ F^3 = \beta \partial^3 E^3 \]

\[ F^y = \alpha E^y \partial_y E^y \bigg|_{y=R} \]

\[ = -\alpha \frac{V_0}{c} \frac{I/c}{2\pi R} \frac{I/c}{2\pi} \frac{V_0}{c} \frac{1}{\partial_y y} \bigg|_{y=R} \]

\[ F^y = -\alpha \frac{(I/c)^2}{R^3} \frac{(V_0)^2}{(2\pi)^2} \frac{1}{c^2} \]

(\textit{Eq. 4})

We can understand it intuitively as follows:

The charge carriers in the metal experience a force:

\[ 0 \rightarrow \]

\[ \rightarrow I_o \]

The force \( q \vec{V} \times \vec{B} / c \) is down for plusses and up for minus. This polarsizes the sphere:

\[ 0 \rightarrow \]

\[ \rightarrow I_o \]
The plus charges then experience a slightly larger force down because they are closer to the wire than the negative charges. The net force is

$$F_{\text{net}} \sim \left( \frac{Q v B}{c} \right)_{\text{larger}} - \left( \frac{Q v B}{c} \right)_{\text{smaller}} \ (-\hat{y})$$
$$\sim \frac{Q v}{c} \left( -\alpha \frac{\partial B}{\partial y} \right) (-\hat{y})$$

We can estimate the induced charge $Q$. The induced charge $Q$ is such that the electrostatic attraction balances the Lorentz force

$$\frac{Q^2}{\alpha^2} \sim \frac{Q v B}{c}$$

$$Q \sim \frac{a^2 v B}{c}$$

So

$$F_{\text{net}} \sim \frac{a^3 v^2}{c} B \left( -\frac{\partial B}{\partial y} \right) (-\hat{y})$$

This is the order of magnitude of Eq. (2) on the previous page, $\alpha \equiv 4\pi a^3$ is the polarizability.
Radiation from a kick

a) To first order the particles motion is constant

\[ z = v_0 t \]

Then

\[ \frac{dp^x}{dt} = F_0 \sin(k_0 vt) \]

\[ \frac{dx}{dt} = \frac{c^2 p^x}{E} \]

\[ \frac{dx}{dt} \approx c^2 \frac{p^x(t)}{E} \]

\[ \frac{d^2 x}{dt^2} = \frac{c^2 \dot{p}^x(t)}{E} = \frac{F_0}{\delta m} \sin(k_0 vt) \]

b) Then

\[ \frac{dP(t)}{d\Omega} = \frac{q^2}{16 \pi^2 c^3} \frac{(n \times (\hat{n} \cdot \hat{v})) \times \hat{v}^2}{(1 - n \cdot \beta)^5} \]

For directly forward \( \vec{\beta} = \beta \hat{n}, \quad 1 - n \cdot \beta = 1 - \beta \)
Then

\[ \frac{dP}{d\Omega} (T) = \frac{q^2}{16 \pi^2 c^3} \frac{1}{(1-\beta)^5} \left( \frac{\vec{n} \times \vec{\gamma} \times \vec{a}}{c^3} \right)^2 (1-\beta)^2 \]

\[ \frac{q^2}{16 \pi^2 c^3} \frac{1}{(1-\beta)^3} a^2 \]

\[ = \frac{q^2}{16 \pi^2 c^3} \left( \frac{2 \gamma^2}{c^3} \right)^3 \left( \frac{F_0}{\gamma m} \right)^2 \sin^2(k_0 vt) \]

\[ \frac{dW}{dT d\Omega} = \frac{q^2}{2 \pi^2 c^3} \gamma^4 \left( \frac{F_0}{m} \right)^2 \sin^2(k_0 vt) \]

Integrating over time \( \int_{\text{period}} dT \sin^2(k_0 vt) = \frac{1}{2} \) (period)

\[ \frac{dW}{d\Omega} = \frac{q^2}{2 \pi^2 c^3} \gamma^4 \left( \frac{F_0}{m} \right)^2 \frac{1}{2} \frac{2\pi}{k_0 v} \]
c) To find the total we use the Larmor

\[ \frac{dW}{dT} = \frac{e^2}{4\pi} \frac{2}{3} \gamma^4 q_1^2 \]

\[ \frac{dW}{dT} = \frac{e^2}{4\pi} \frac{2}{3} \gamma^4 \left( \frac{F_o}{\partial m} \right)^2 \sin^2 (k_o v T) \]

Integrating over time from

\[ T = -\frac{\Pi}{k_o v} \ldots \frac{\Pi}{k_o v} \]

(This is the time that it interacts with the force)

\[ \frac{dW}{dT} = \frac{e^2}{4\pi} \frac{2}{3} \gamma^2 \left( \frac{F_o}{m} \right)^2 \frac{1}{2} \frac{2\Pi}{k_o v} \]
d) To find the frequency spectrum we use

$$E(\omega, r) = \frac{q}{\pi \omega r c^2} \int_{-\infty}^{\infty} d\tau e^{i\omega(T - \frac{\tau}{c} + \frac{\tau}{c} - \beta)}$$

$$\left[ \frac{\hat{n} \cdot (\hat{n} - \hat{\beta}) \cdot \hat{a}}{(1 - \hat{n} \cdot \hat{\beta})^2} \right]$$

Now, \( \hat{n} = \hat{z} \) and \( \hat{n} - \hat{\beta} = (1 - \beta) \hat{z} \), \( (1 - n \cdot \beta) = 1 - \beta \)

and \( \Gamma_k(T) = \sqrt{T} \) so

$$E(\omega, r) = \frac{q}{\pi \omega r c^2} \int_{-\infty}^{\infty} d\tau e^{i\omega (1 - \beta) \tau}$$

$$\frac{F_0 \sin(k \sqrt{T})}{\delta m} \left( \frac{-\xi}{1 - \beta} \right)$$

So

$$2\pi dW = c |\tilde{E}(\omega, r)|^2$$

d\omega d\Omega

using \( \left( \frac{F}{\delta m (1 - \beta)} \right)^2 = 4 \xi^2 \left( \frac{F_0}{m} \right)^2 \)
We have

\[
2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2 \gamma^2}{4 \pi^2 c^3} \left( \frac{E}{m} \right)^2 |I|^2
\]

Where

\[
I = \int_{-\infty}^{\infty} e^{i \omega (1-\beta) T} \sin (k_v VT)
\]

Define \( z = k_v VT \) \( u = \frac{\omega (1-\beta)}{k_v V} \)

\[
I = \frac{1}{k_v V} \int_{-\pi}^{\pi} e^{i u z} \sin (z)
\]

\[
= 2i \frac{\sin (\pi u)}{k_v V (1-u^2)}
\]

Then

\[
2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2 \gamma^2}{4 \pi^2 c^3} \left( \frac{E}{m} \right)^2 \frac{\gamma^2}{(k_v V)^2} \left[ \frac{\sin (\pi u)}{(1-u^2)} \right]^2
\]

\( u = \frac{\omega (1-\beta)}{k_v V} \)
We can check this result by integrating over \( w \) or \( u \):

\[
\int_{-\infty}^{\infty} dw \cdots = \int_{-\infty}^{\infty} du \frac{k_0 V}{2\pi (1-\beta)} \cdots
\]

Then by convolution theorem (notice Eq \( \xi \) is a Fourier transform):

\[
\int_{-\infty}^{\infty} du \frac{1}{2\pi} \left( \frac{\sin \pi u}{(1 - u^2)} \right)^2 = \int_{-\pi}^{\pi} dx \sin^2 x = \pi
\]

So

\[
\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dw \frac{2\pi}{2\pi} \frac{dW}{dw} dw d\Omega
\]

\[
= \frac{q^2}{4\pi^2 c^3} \left( \frac{F}{m} \right)^2 \gamma^2 \frac{\pi k_0 V}{(1-\beta)} \uparrow \frac{1}{1-\beta} = 2\gamma^2
\]

\[
\frac{dW}{d\Omega} = \frac{q^2}{2\pi c^3} \left( \frac{F}{m} \right)^2 \gamma 4 \uparrow \frac{1}{1-\beta}
\]

This agrees with part (b).
e) The typical frequency is

\[ \omega \sim \frac{k_0 v \sim \gamma^2 k_0 v}{(1-\beta)} \]

This is expected. The formation time (the time that the particle is accelerated) is of order \( \frac{1}{k_0 v} \) (The duration of the acceleration is \( 2\pi/k_0 v \)).

Then if a wave was formed/emitted over a time \( \Delta T \), then it arrives at detector over a time \( \Delta t \)

\[ \Delta t = \frac{\Delta T}{\frac{\Delta T}{\Delta t} = \Delta T \frac{1}{(1-n_0 \beta)}} \]

So the duration of the pulse seen by the detector is:

\[ \Delta t \sim \frac{2\pi}{k_0 v (1-\beta)} \]

So by the uncertainty principal, the typical frequency is

\[ \omega \sim \frac{1}{\Delta t} \sim \frac{k_0 v \sim \gamma^2 k_0 v}{(1-\beta)} \]