Grad, Div, Curl, and Laplacian

CARTESIAN $d\boldsymbol{\ell} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ $\quad d^3r = dxdydz$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\psi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

CYLINDRICAL $d\boldsymbol{\ell} = d\rho\hat{\boldsymbol{\rho}} + \rho d\phi\hat{\boldsymbol{\phi}} + dz\hat{\mathbf{z}}$ $\quad d^3r = \rho d\rho d\phi dz$

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right]\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

SPHERICAL $d\boldsymbol{\ell} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}$ $\quad d^3r = r^2\sin\theta dr d\theta d\phi$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Figure 1: Grad, Div, Curl, Laplacian in Cartesian, cylindrical, and spherical coordinates. Here ψ is a scalar function and $\mathbf{A}$ is a vector field.

Vector Identities

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

Integral Identities

$$\int_V d^3r\,\nabla\cdot\mathbf{A} = \int_S dS\,\hat{\mathbf{n}}\cdot\mathbf{A}$$

$$\int_V d^3r\,\nabla\psi = \int_S dS\,\hat{\mathbf{n}}\psi$$

$$\int_V d^3r\,\nabla\times\mathbf{A} = \int_S dS\,\hat{\mathbf{n}}\times\mathbf{A}$$

$$\int_S dS\,\hat{\mathbf{n}}\cdot\nabla\times\mathbf{A} = \oint_C d\boldsymbol{\ell}\cdot\mathbf{A}$$

$$\int_S dS\,\hat{\mathbf{n}}\times\nabla\psi = \oint_C d\boldsymbol{\ell}\,\psi$$

Figure 2: Vector and integral identities. Here ψ is a scalar function and $\mathbf{A}, \mathbf{a}, \mathbf{b}, \mathbf{c}$ are vector fields.

Problem 1. An uncharged rotor

Two equal and opposite charges are attached to the ends of a rod of length s as shown below. The rod rotates (non-relativistically) counter-clockwise in the xy plane with angular speed $\omega_o = ck_o$.

(a) Explicitly determine the components of the radiation electric field as a function of time in the far field as a function of r, θ in the xz plane (see below). The answer should take the form

$$\boldsymbol{E}(r,\theta,t) = E_\theta(r,\theta,t)\,\widehat{\boldsymbol{\theta}} + E_\phi(r,\theta,t)\,\widehat{\boldsymbol{\phi}}\,. \tag{1}$$

Why is there no $\widehat{\boldsymbol{r}}$ component in Eq. (1)? Hint: it may be helpful (but not essential) to express the position of the ends of the rod using a complex notation, e.g. the position of the plus charge is $\boldsymbol{r}_+(t) = (s/2)\,(\widehat{\boldsymbol{x}} + i\widehat{\boldsymbol{y}})\,e^{-i\omega_o t}$.

(b) Write out the real electric field on the x and z axes in cartesian coordinates. Identify the state of polarization that is observed in the two cases and give a physical explanation for the observed polarization.

(c) Determine the time averaged rate at which energy is radiated per unit solid angle.

(d) After spinning for a long time (from time $t = -\infty \ldots 0$), the rotating rod abruptly stops when the azimuthal angle $\phi_o = 0$. For a detector placed in the far field on the z axis ($\theta = \phi = 0$) determine the energy per frequency per solid angle

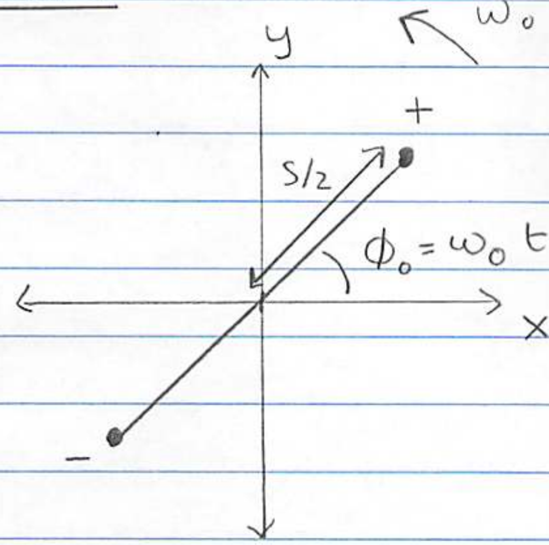
$$(2\pi)\frac{dW}{d\omega d\Omega}\,, \tag{2}$$

and the yield of photons per frequency interval

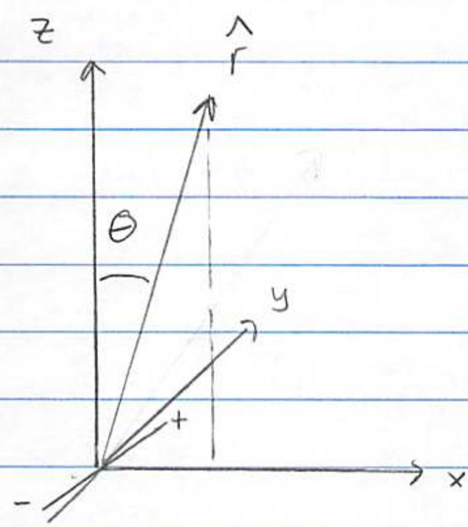
$$\frac{dN}{d\omega d\Omega}\,. \tag{3}$$

Problem 1

Top View



3D view



Problem 2.   Potential difference across a cylinder

A neutral thin dielectric cylindrical shell of radius a and thickness t (with $t \ll a$) rotates non-relativistically with constant angular velocity ω_o with $\omega_o a/c \ll 1$ (see below). The cylindrical shell sits in a constant homogeneous magnetic field directed along the z axis, $\boldsymbol{B} = B_o \,\widehat{\boldsymbol{z}}$ (see below). A potential difference of ΔV is observed between the inside and outsides of the cylindrical shell as shown below. The cylinder has dielectric constant $\epsilon = 1 + \chi$ with $\chi \ll 1$.

(a) Recall that the vector potential of a constant magnetic field is $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r}$. By making a Lorentz transformation of the four potential A^μ (in the Lorentz gauge) determine the potential $\underline{A}^\mu$ in the co-rotating frame of the cylinder (i.e. a frame moving with the walls of the cylinder)

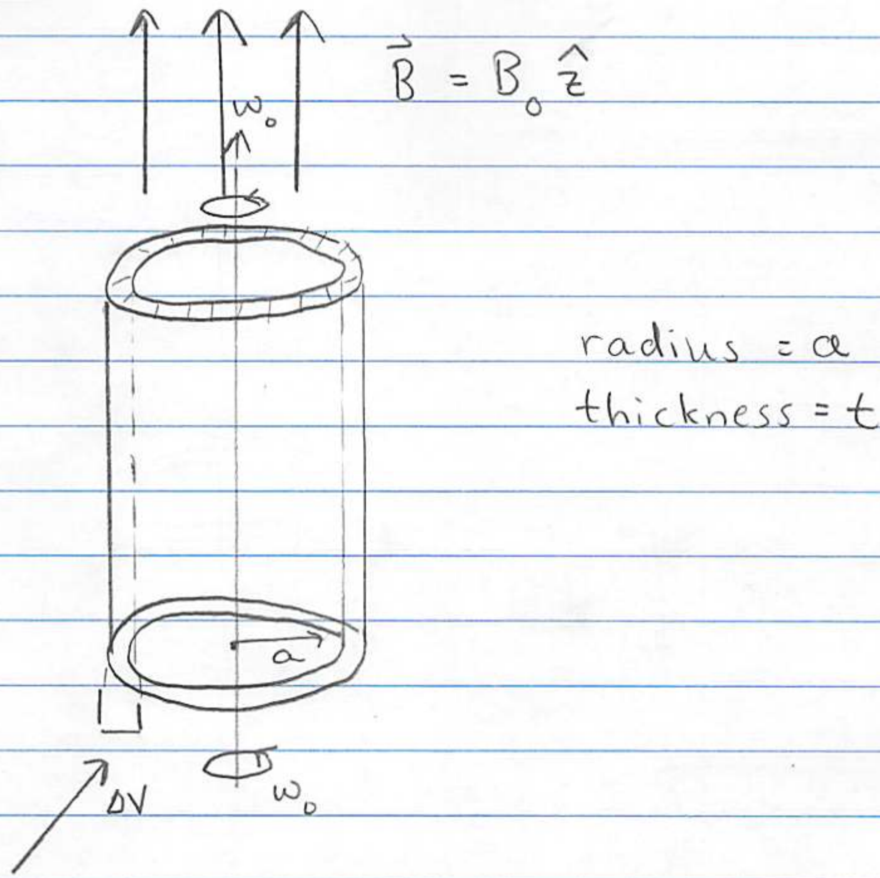
(b) In a co-rotating frame determine the electric field experienced by the cylinder? Is your electric field consistent with the gauge potential $\underline{A}^\mu$ of part (a) ?

(c) In the co-rotating frame determine the charge density on the surface of cylinder and make a sketch

(d) Qualitatively explain the potential difference ΔV in the lab frame.

(e) Quantitatively determine the potential difference ΔV in the lab frame. Indicate the direction of the (weak) electric field in the lab frame by making a sketch.

Problem 2



measure potential difference between the inside and the outside of cylinder ΔV

Problem 3. Radiation from a proper acceleration

(a) Write down the covariant action of a relativistic point particle coupled to a gauge field A^μ and show that the action is gauge invariant.

(b) Using the action of part (a) determine the covariant equations of motion for a relativistic point particle in an electromagnetic field. Use the covariant equation of motion to show that $U_\mu U^\mu = \text{const}$.

(c) Set $c = 1$ for simplicity. Now consider an ultra-relativistic relativistic positron of positive charge q and mass m traveling with velocity $v_o \equiv \tanh y_o$ in the negative x-direction from positive infinity (see below). (Note that $\gamma_o = \cosh y_o$ and $\gamma_o v_o = \sinh y_o$.) At $x = 0$ the particle enters a semi-infinite region ($x < 0$) of homogeneous electric field directed in the positive x-direction, $\boldsymbol{E} = E\,\widehat{\boldsymbol{x}}$. The particle experiences a constant force, decelerates to a momentary stop, and is finally re-accelerated to its original speed (but in the opposite direction) by the time it leaves the electric field again (see below).

  (i) Determine the position $x(\tau)$ and the four velocity $u^\mu(\tau) = dx^\mu/d\tau$ as a function of proper time τ while the particle is in the electric field. Also determine the relation between the proper time τ and t.

  (ii) How long (in time) does the particle remain in the electric field, and how far to the left of $x = 0$ (d_{max} in the figure below) does the particle penetrate into the field?

Hint: Recall the properties hyperbolic functions

$$\cosh y = \cos(iy) = \frac{e^y + e^{-y}}{2} \qquad \sinh(y) = -i\sin(iy) = \frac{e^y - e^{-y}}{2} \tag{4}$$

and its properties

$$\frac{d}{dy}\cosh(y) = \sinh y \qquad \frac{d}{dy}\sinh(y) = \cosh(y) \qquad \cosh^2 y - \sinh^2 y = 1 \tag{5}$$

You may find these relations useful in integrating the equations of motion, i.e. expressing the rapidity y as a function of τ.

(d) What is the energy lost to radiation during the relativistic motion of part (c)? Express your answer as a dimensionless (order unity) integral which you may leave unevaluated.

(e) (**extra-credit**) For $\gamma_o \sim 10$ estimate the electric field (in Volts/meter) where the energy lost equals $\sim 1\%$ of the initial energy.

Problem 3