C Fourier Series and other eigenfunction expansions

We will often expand a function in a complete set of eigen-functions. Many of these eigen-functions are traditionally not normalized. Using the quantum mechanics notation we have

$$|F\rangle = \sum_n F_n \frac{1}{C_n} |n\rangle \qquad \text{where} \qquad F_n = \langle n|F\rangle \qquad \text{and} \qquad \langle n_1|n_2\rangle = C_{n_1}\delta_{n_1 n_2} \tag{C.1}$$

or more prosaically:

$$F(x) = \sum_n F_n \frac{1}{C_n} [\psi_n(x)] \,, \tag{C.2}$$

$$F_n = \int dx\, \psi_n^*(x)\, F(x)\,, \tag{C.3}$$

$$\int dx\, \left[\psi_{n_1}^*(x)\right] \left[\psi_{n_2}(x)\right] = C_{n_1}\delta_{n_1 n_2}\,. \tag{C.4}$$

We require that the functions are complete (in the space of functions which satisfy the same boundary conditions as F) and orthogonal

$$\sum_n \frac{1}{C_n} |n\rangle \langle n| = I\,, \qquad \text{or} \qquad \sum_n \frac{1}{C_n} \psi_n(x)\psi_n^*(x') = \delta(x - x')\,. \tag{C.5}$$

In what follows we show the eigen-function in square brackets

(a) A periodic function $F(x)$ with period L is expandable in a Fourier series. Defining $k_n = 2\pi n/L$ with n integer:

$$F(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty} \left[e^{ik_n x}\right] F_n \tag{C.6}$$

$$F_n = \int_0^L dx\, \left[e^{-ik_n x}\right] F(x) \tag{C.7}$$

$$\int_0^L dx\, [e^{-ik_n x}]\, [e^{ik_{n'} x}] = L\,\delta_{nn'} \tag{C.8}$$

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} e^{ik_n(x-x')} = \sum_m \delta(x - x' + nL) \tag{C.9}$$

(b) A square integrable function in one dimension has a Fourier transform

$$F(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[e^{ikz}\right] F(k) \tag{C.10}$$

$$F(k) = \int_{-\infty}^{\infty} dz \left[e^{-ikz}\right] F(z) \tag{C.11}$$

$$\int_{-\infty}^{\infty} dz\, e^{-iz(k-k')} = 2\pi\delta(k-k') \tag{C.12}$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(z-z')} = \delta(z-z') \tag{C.13}$$

(c) A regular function on the sphere (θ, ϕ) can be expanded in spherical harmonics

$$F(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [Y_{\ell m}(\theta,\phi)]\, F_{\ell m} \tag{C.14}$$

$$F_{\ell m} = \int d\Omega\, [Y^*_{\ell m}(\theta,\phi)]\, F(\theta,\phi) \tag{C.15}$$

$$\int d\Omega\, [Y^*_{\ell m}(\theta,\phi)]\, [Y_{\ell' m'}(\theta,\phi)] = \delta_{\ell\ell'}\delta_{mm'} \tag{C.16}$$

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [Y_{\ell m}(\theta,\phi)]\, [Y^*_{\ell m}(\theta',\phi')] = \delta(\cos\theta - \cos\theta')\delta(\phi-\phi') \tag{C.17}$$

(d) When expanding a function on the sphere with azimuthal symmetry, the full set of $Y_{\ell m}$ is not needed. Only $Y_{\ell 0}$ is needed. $Y_{\ell 0}$ is related to the Legendre Polynomials. We note that

$$Y_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta) \tag{C.18}$$

A function $F(\cos\theta)$ between $\cos\theta = -1$ and $\cos\theta = 1$ can be expanded in Legendre Polynomials.

$$F(\cos\theta) = \sum_{\ell=0}^{\infty} F_\ell\, \frac{2\ell+1}{2}\, [P_\ell(\cos\theta)] \tag{C.19}$$

$$F_\ell = \int_{-1}^{-1} d(\cos\theta)\, [P_\ell(\cos\theta)]\, F(\cos\theta) \tag{C.20}$$

$$\int_{-1}^{1} d(\cos\theta)\, [P_\ell(\cos\theta)]\, [P_{\ell'}(\cos\theta)] = \frac{2}{2\ell+1}\delta_{\ell\ell'} \tag{C.21}$$

$$\sum_{\ell=0}^{\infty} \frac{2\ell+1}{2}\, [P_\ell(\cos\theta)]\, [P_\ell(\cos\theta')] = \delta(\cos\theta - \cos\theta') \tag{C.22}$$

(e) A function, $F(\rho)$ on the half line $\rho = [0,\infty]$, which vanishes like ρ^m as $\rho \to 0$ can be expanded in Bessel functions. This is known as a Hankel transform and arises in cylindrical coordinates

$$F(\rho) = \int_0^{\infty} kdk\, [J_m(k\rho)]\, F_m(k) \tag{C.23}$$

$$F_m(k) = \int_0^{\infty} \rho d\rho\, [J_m(k\rho)]\, F(\rho) \tag{C.24}$$

$$\int_0^{\infty} \rho d\rho\, [J_m(\rho k)]\, [J_m(\rho k')] = \frac{1}{k}\delta(k-k') \tag{C.25}$$

$$\int_0^{\infty} kdk\, [J_m(\rho k)]\, [J_m(\rho' k)] = \frac{1}{\rho}\delta(\rho-\rho') \tag{C.26}$$