Problem 1. Proper acceleration

A particle of mass $m$, starting at rest at time $t = 0$ and $x = 0$ in the lab frame, experiences a constant acceleration, $a$, in the $x$-direction in its own rest frame.

(a) The acceleration four vector

$$ A^\mu \equiv \frac{d^2x^\mu}{d\tau^2} $$

is specified by the problem statement. What are the four components of the acceleration four vector in the rest frame of the particle and in the lab frame. What is the acceleration, $d^2x/dt^2$, in the lab frame.

(b) Show that the position of the particle as a function of time can be parameterized by a real number $p$

$$ x = \frac{c^2}{a}[\cosh(p) - 1] $$

where $p$ is related to the time $t$ through the equation:

$$ ct = \frac{c^2}{a}\sinh(p) $$

(c) Show that the parameter $p$ is proportional proper time, $p = \frac{a}{c} \tau$.

(d) The rapidity of a particle, $Y$, is defined by its velocity

$$ \frac{v}{c} \equiv \tanh(Y) $$

where $v = dx/dt$. Show that the four velocity $u^\mu = dx^\mu/d\tau$ is related to the rapidity through the hyperbolic relations.

$$ (u^0/c, u^1/c) = (\cosh(Y), \sinh(Y)) $$

(e) Show that $Y = a\tau/c$

**Remark:** We see that the rapidity of the particle increases linearly with proper time during proper acceleration.

(f) (Optional) If the particle has a constant decay rate in its own frame of $\Gamma$, show that the probability that the particle survives at late time $t$ is approximately

$$ \left( \frac{2\alpha t}{c} \right)^{-\Gamma c/a} $$
Constant Acceleration

\[ A^\mu = \frac{d^2 x^\mu}{d\tau^2} \]

a) In the rest frame of particle

\[ A^\mu = \begin{pmatrix} 0 \\ \alpha/c^2 \end{pmatrix} \]

Then in the lab frame

\[ \begin{pmatrix} A^0 \\ A^1 \end{pmatrix} = \begin{pmatrix} \gamma u & \gamma \beta u \\ \gamma \beta u & \gamma \tau \end{pmatrix} \begin{pmatrix} 0 \\ \alpha/c^2 \end{pmatrix} \]

\[ \begin{align*}
\frac{dx^0}{d\tau^2} &= \gamma \beta \gamma u \alpha/c^2 \\
\frac{dx^1}{d\tau^2} &= \gamma u \alpha/c^2
\end{align*} \]

Using \( d\tau = \gamma dt \)

\[ \delta^n u \frac{du}{dt} = \gamma u \alpha/c^2 \]

\[ \frac{du}{dt} = \alpha/c^2 \]
Using

\( \ddot{u} = \gamma v \)

\[
\frac{d\ddot{u}}{dt} = \frac{1}{\gamma^2} \left[ (1 - v^2)^{-3/2} \left( +2 \dot{v} (1 - v^2)^{1/2} \right) \right] \dot{v} + \frac{\dot{v}}{\gamma} \dot{v} = \gamma (\gamma^2 v^2 + 1) \dot{v}
\]

\[ \frac{d\ddot{u}}{dt} = \gamma^3 \dot{v} \]

So

\[
\dot{v} = \frac{\alpha / c^2}{\gamma^3}
\]

\[ \ddot{u} = \gamma \dot{v} \]

b) From \( \frac{d\ddot{u}}{dt} = \alpha / c^2 \) note \( \gamma = \frac{1}{\sqrt{1 - (u/c)^2}} = \frac{1 + \dot{u}^2}{c^2} \)

\[ \ddot{u} = \alpha / c^2 \dot{t} \]

\[ \alpha = \frac{\alpha}{c^2} \]

\[
\sqrt{1 + (\alpha / c^2)^2} \frac{d\lambda}{c^2} = \alpha / c^2 \dot{t}
\]

\[ \frac{d\lambda}{c^2} = \frac{\dot{t}}{\sqrt{1 + (\alpha / c^2)^2}} \]
Doing the integral

\[ x = \int \frac{dt}{\sqrt{1 + (at)^2 / c^2}} \]

let \( \frac{at}{c} = \sinh p \)

\[ d(at) = \cosh y \, dy \]

\[ \cosh^2 y - \sinh^2 y = 1 \]

\[ 1 - \tanh^2 y = \text{sech}^2 y \]

\[ x = \frac{c^2}{\alpha} \int \frac{\cosh p \, dp}{\sqrt{1 + \sinh^2 p}} \]

\[ x = \frac{c^2}{\alpha} \int dp \, \sinh p \]

\[ x = \frac{c^2}{\alpha} \cosh p + C \]

where \( at = \sinh y \). Fixing the constant so at \( t = 0, \, x = 0 \)

\[ x = \frac{c^2}{\alpha} (\cosh p - 1) \]

\[ ct = \frac{c^2}{\alpha} \sinh p \]
From the solution

c) \[ dx^0 = \frac{c^2}{a} \cosh p \, dp \]

\[ dx^1 = \frac{c^2}{a^2} \sinh p \, dp \]

So
\[ c^2 d\tau^2 = -dx^0 + dx^1^2 \]

\[ c^2 d\tau^2 = \left( \frac{c^2}{a} \right)^2 \left( \cosh^2 p - \sinh^2 p \right) (dp)^2 \]

\[ c \, d\tau = \frac{c^2}{a} \, dp \]

Integrating
\[ \frac{a \, \tau}{c} = p + \text{const} \leftarrow \text{the const can be set to zero} \]

\[ \frac{a \, \tau}{c} = p \]
d) The rapidity

\[ \nu = \tanh y \]

Then

\[ \frac{\mathbf{u}^\nu}{c} = (\gamma, \gamma \mathbf{u}^\nu) \]

\[ \frac{\mathbf{u}^\nu}{c} = \left( \frac{1}{\sqrt{1 - \tanh^2 y}}, \frac{\tanh y}{\sqrt{1 - \tanh^2 y}} \right) \]

using:

\[ \cosh^2 - \sinh^2 = 1 \implies \frac{1}{\sqrt{1 - \tanh^2 y}} = \cosh y \]

So

\[ \frac{\mathbf{u}^\nu}{c} = (\cosh y, \sinh y) \]
e) Then using

\[ \frac{dx^0}{dt} = \frac{d}{dt} \left[ \frac{c^2 \sinh \left( \frac{a \cdot t}{c} \right)}{a} \right] \]

\[ \frac{dx^1}{dt} = \frac{d}{dt} \left[ \frac{c^2 \cosh \left( \frac{a \cdot t}{c} \right)}{a} \right] \]

\[
\begin{align*}
\frac{dx^0}{dt} &= c \cosh \left( \frac{a \cdot t}{c} \right) \\
\frac{dx^1}{dt} &= c \sinh \left( \frac{a \cdot t}{c} \right)
\end{align*}
\]

Comparison shows \( u \)

\[
\nabla \cdot \mathbf{u} = \frac{a \cdot t}{c}
\]
The probability of surviving is
\[ e^{-\Pi \tau} \]

Using
\[ ct = \frac{c^2}{a} \sinh \left( \frac{a + \tau}{c} \right) \]

And late times where \( \sinh x = \frac{e^x - e^{-x}}{2} \approx e^x \)

we have
\[ ct = c^2 \frac{e^{\frac{at}{c}}}{2a} \]

or
\[ \frac{at}{c} \approx \frac{\ln 2at}{c} \Rightarrow \tau \approx \frac{c}{a} \frac{\ln 2at}{c} \]

So
\[ e^{-\Pi \tau} \approx e^{-\frac{\ln 2at}{c}} \approx \left( \frac{2at}{c} \right)^{-\frac{c}{\alpha}} \]
Problem 2. The stress tensor from the equations of motion

In class we wrote down energy and momentum conservation in the form

\[ \frac{\partial \Theta_{\text{mech}}^{\mu\nu}}{\partial x^\mu} = F_\nu^\rho \frac{J^\rho}{c} \]  

(6)

Where the \( \nu = 0 \) component of this equation reflects the work done by the E&M field on the mechanical constituents, and the spatial components \( (\nu = 1, 2, 3) \) of this equation reflect the force by the E&M field on the mechanical constituents.

(a) Verify that

\[ F_\nu^\rho \frac{J^\rho}{c} = \begin{cases} \left( J/c \cdot E \right) \quad & \nu = 0 \\
\rho E^j + \left( J/c \times B \right)^j \quad & \nu = j \end{cases} \]  

(7)

(b) (Optional) Working within the limitations of magnetostatics where

\[ \nabla \times B = \frac{J}{c} \quad \nabla \cdot B = 0 \]  

(8)

show that the magnetic force can be written as the divergence of the magnetic stress tensor, \( T_B^{ij} = -B^i B^j + \frac{1}{2} \delta^{ij} B^2 \):

\[ \left( \frac{J}{c} \times B \right)^j = -\partial_i T_B^{ij} \]  

(9)

(c) Consider a solenoid of infinite length carrying current \( I \) with \( n \) turns per length, what is the force per area on the sides of the solenoid.

(d) Using the equations of motion in covariant form

\[ -\partial_\mu F^{\mu\rho} = \frac{J^\rho}{c} \]  

(10)

and the Bianchi Identity

\[ \partial_\mu F_{\sigma\rho} + \partial_\sigma F_{\rho\mu} + \partial_\rho F_{\mu\sigma} = 0 \]  

(11)

show that

\[ F_\nu^\rho \frac{J^\rho}{c} = -\frac{\partial}{\partial x^\mu} \Theta_{\text{em}}^{\mu\nu} \]  

(12)

where

\[ \Theta_{\text{em}}^{\mu\nu} = F^{\mu\rho} F_\rho^\nu + g^{\mu\nu} \left( -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right) \]  

(13)

Hint: use the fact that \( F^{\mu\rho} \) is anti-symmetric under interchange of \( \mu \) and \( \rho \).

(e) (Optional) Verify by direct substitution, using \( F^{ij} = \epsilon^{ijk} B_k \), that if there is no electric field that

\[ \Theta^{ij} = T_B^{ij} \]  

(14)
The stress tensor from EOM

\[ F_\rho^\alpha J^\rho = F_\xi^\alpha J^\xi = \frac{E \cdot J}{c} \]  

(1)

Similarly,

\[ F_\rho^i J^\rho = F_\xi^i J^\xi + F_\delta^i J^\delta \]

\[ = E^i_\rho \epsilon^{ijk} B_k J^\delta \]

(2)

\[ = E^i_\rho + \left( \frac{\tau^i B}{c} \right) \text{'} \]  

(3)

b) In magneto-statics

\[ \mathcal{E}^\rho \mathcal{B} = (\frac{\mathcal{J} \times \mathcal{B}}{c}) \text{'} \]

\[ = \mathcal{E}^{jlm} \epsilon_{lmn} B\text{'}_n \text{ using } \mathcal{J}_x = \mathcal{E}_{lno} \partial_n B_o \]

\[ = \mathcal{E}^{jlm} \epsilon_{lmn} (\partial_n B_o) B\text{'}_m \]

\[ \mathcal{J}_x = \mathcal{E}^{jlm} \epsilon_{lmn} (\partial_n B_o) B\text{'}_m \]

\[ = \left( \delta^{mn} \delta^{\rho \delta} - \delta^{mo} \delta^{\rho n} \right) (\partial_n B_o) B\text{'}_m \]
\[ f^\dot{a} = - \frac{\partial}{\partial x^n} (T^n a) \]

\[ T^n a = - B^n B^\dot{a} + \frac{\delta^n a}{2} B^2 \]

\[ f^\dot{a} = + B^n \partial_n B^\dot{a} - (\partial \delta^m) B^m \]

\[ = + B^n \partial_n B^\dot{a} - \frac{1}{2} \frac{\delta_{\alpha n} \partial_n (B^2)}{2} \]

Use \[ \partial_n B^n = 0 \]

\[ = \frac{\partial}{\partial x^n} \left[ B^n B^\dot{a} - \frac{1}{2} \delta_{\alpha n} \partial_n (B^2) \right] \]

\[ = \frac{\partial}{\partial x^n} (T^n a) \]

---

(c) Covariant Treated

\[ f^\nu = F^\nu_{\rho} J^\rho \]

\[ f^\nu = - F^\nu_{\rho} \partial_{x^\rho} F^{\mu \rho} \]

\[ = - \frac{\partial}{\partial x^\rho} F^{\mu \rho} F^\nu_{\rho} + \left( g^{\nu \sigma} \frac{2}{\partial x^\mu} F_{\sigma \rho} \right) F^{\mu \rho} \]
Part c

Using ampere's law:

\[ B \cdot L = n L \frac{I_0}{c} \]

Inside:

\[ B = n \frac{I_0}{c} \]

Outside:

\[ B = 0 \]

Then:

\[ T_{pp} = \frac{\text{Force}}{\text{Area}} = \frac{\text{Force in } p\text{-direction}}{\text{Area in } p\text{-direction}} \]

Then:

\[ B = 0 \rightarrow B = 0 \]

Net force:

\[ \text{net force} = T_{in}^{pp} - T_{out}^{pp} = -n \left( T_{out}^{pp} - T_{in}^{pp} \right) = T_{in}^{pp} \]

So:

\[ T_{in}^{pp} = -B_{in}^{P} B_{in}^{P} + \delta_{PP} B^2 \]

\[ T_{in}^{pp} = \frac{1}{2} \left( \frac{n I_0}{c} \right)^2 \]
while force law gives twice this

\[ \frac{dF}{c} = dI \cdot dB \] (wrong?)

\[ = \frac{nI_o \, dh \, dl \, B}{c} \]

\[ \int dI = nI_o \, dt \]

So

\[ \frac{\text{Force}}{\text{Area} \cdot dh \, dl} = \frac{nI_o}{c} B \]

The question is what to take for \( B \). \( B_{in} = nI_o \)
or \( B_{out} = 0 \). The stress tensor gives the answer

\[ B = B_{in} + B_{out} = \frac{1}{2} B_{in} \]
**Part D - Covariant Treatment**

\[ f^\nu = F^\nu_\rho \frac{j^\rho}{c} \]

\[ f^\nu = - F^\nu_\rho \frac{\partial}{\partial x^\mu} F^{\mu\rho} \]

\[ = - \frac{\partial}{\partial x^\mu} (F^{\mu\rho} F^{\nu}_\rho) \]

Then

We

Continue

on

next page
Using

$$
\partial_\mu F_{\nu\rho} + \partial_\sigma F_{\rho\mu} + \partial_\rho F_{\mu\sigma} = 0
$$

$$
[\partial_\nu F_{\rho\sigma} - \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho}] = 0
$$

We see since $F^{\mu\rho}$ is antisymmetric,

$$
F^{\mu\rho} (\partial_\nu F_{\rho\sigma}) = \frac{1}{2} F^{\mu\rho} (\partial_\nu F_{\rho\sigma} - \partial_\sigma F_{\rho\nu})
$$

$$
= \frac{-1}{2} \partial_\nu F_{\rho\sigma}
$$

$$
= \frac{1}{2} \partial_\rho F_{\mu\sigma}
$$

That

$$
f^\nu = -\frac{1}{2} \left( F^{\mu\rho} F^\nu_{\rho} \right) + g^{\nu\sigma} \frac{1}{2} F^{\mu\rho} \partial_\nu F_{\sigma\rho}
$$

relabel, indices

$$
\mu = -\frac{1}{2} \left[ F^{\mu\rho} F^\nu_{\rho} + g^{\mu\nu} (-1 F^{\alpha\beta} F_{\alpha\beta}) \right]
$$

$$
f^\nu = -\frac{1}{2} \partial_\mu \Theta_{\mu\nu}
$$

$$
\Theta_{\mu\nu} = F^{\mu\rho} F^\nu_{\rho} + g^{\mu\nu} (-1 F^2)
$$
d) For \( E=0, \ F^{10}=0, \ -F_{\frac{1}{2}}^{2} = -\frac{1}{2} B^2 \), and \\
\[ \Theta^{i'y} = F^{i'k} F_{k}^{y} + \delta^{i'y} \left( -\frac{1}{2} B^2 \right) \]

So

\[ \Theta^{i'y} = \varepsilon^{ikm} B_k \varepsilon_{l'ym} B_m + \delta^{i'y} \left( -\frac{1}{2} B^2 \right) \]

\[ = B_k B_m \left( \varepsilon^{lk'i} \varepsilon_{l'm'y} \right) + \delta^{i'y} \left( -\frac{1}{2} B^2 \right) \]

\[ = B_k B_m (\delta^{km} \delta^{i'y} - \delta^{ki} \delta^{i'm}) + \delta^{i'y} \left( -\frac{1}{2} B^2 \right) \]

\[ = \left[ -B_{\frac{1}{2}} B^{i'y} + \delta^{i'y} B^2 \right] + \delta^{i'y} \left( -\frac{1}{2} B^2 \right) \]

\[ \Theta^{i'y} = -B_{\frac{1}{2}} B^{i'y} + \delta^{i'y} \left( +\frac{1}{2} B^2 \right) \]
Problem 3. Fields from moving particle

The electric and magnetic fields of a particle of charge $q$ moving in a straight line with speed $v = \beta c$ were given in class. Choose the axes so that the charge moves along the $z-$axis in the positive direction, passing the origin at $t = 0$. Let the spatial coordinates of the observation point be $(x, y, z)$ and define a transverse vector (or impact parameter) $b_{\perp} = (x, y)$, with components $x$ and $y$. Consider the fields and the source in the limit $\beta \rightarrow 1$

(a) First (keeping $\beta$ finite) find the vector potential $A^\mu$ associated with the moving particle using a Lorentz transformation. Determine the field strength tensor $F^{\mu\nu}$ by differentiating $A^\mu$, and verify that you get the same answer as we got in class.

(b) As the charge $q$ passes by a charge $e$ at impact parameter $b$, show that the accumulated transverse momentum transfer (transverse impulse) to the charge $e$ during the passage of $q$ is

$$\Delta p_{\perp} = \frac{eq}{2\pi b_{\perp}^2 c}$$  \hspace{1cm} (15)

(c) Show that the time integral of the absolute value of the longitudinal force to a charge $e$ at rest at an impact parameter $b_{\perp}$ is

$$\frac{eq}{2\pi \gamma b_{\perp} c}$$  \hspace{1cm} (16)

and hence approaches zero as $\beta \rightarrow 1$.

(d) (Optional) Show that the fields of charge $q$ can be written for $\beta \rightarrow 1$ as

$$E = \frac{q}{2\pi \gamma b_{\perp}^2} \frac{b_{\perp}}{b_{\perp}^2} \delta(ct - z), \quad B = \frac{q}{2\pi \gamma} \frac{\hat{v}}{c} \times b_{\perp} \frac{b_{\perp}}{b_{\perp}^2} \delta(ct - z).$$  \hspace{1cm} (17)

(e) (Optional) Show by explicit substitution into the Maxwell equations that these fields are consistent with the 4-vector source density

$$J^\alpha = q\nu^\alpha \delta^2(b_{\perp}) \delta(ct - z)$$  \hspace{1cm} (18)

where $\nu^\alpha = (c, \hat{v})$. 

The fields are
\[
E_{\parallel} = \frac{1}{\pi} \frac{q}{b^2 + \gamma^2 (z-vt)^2} \left[ b^2 + \frac{\gamma^2}{b^2} (z-vt)^2 \right]^{3/2}
\]
\[
E_{\perp} = \frac{1}{\pi} \frac{q}{b^2 + \gamma^2 (z-vt)^2} \left[ b^2 + \frac{\gamma^2}{b^2} (z-vt)^2 \right]^{3/2}
\]
\[
B = \frac{\gamma}{c} \times \mathbf{E}
\]

The transverse force at \( z=0 \) at time \( t \) is
\[
F_{\perp} = e \mathbf{E}_{\perp}(z=0,t)
\]

and
\[
\Delta p_{\perp} = \int_{-\infty}^{\infty} F_{\perp} \, dt = \int_{-\infty}^{\infty} dt \frac{eq}{\pi} \frac{\gamma b}{b^2 + \frac{\gamma^2}{b^2} (z-vt)^2}^{3/2}
\]

Setting \( v = c \), \( \gamma \) large, we have \( \gamma \gg 1 \)

\[
\Delta p_{\perp} = \frac{eq}{\pi c} \int_{-\infty}^{\infty} du \frac{1}{b^2} \left[ \frac{1}{b^2 + u^2} \right]^{3/2}
\]

\[
= \frac{eq}{\pi c} \frac{1}{b^2} \int_{-\infty}^{\infty} du \frac{1}{\left[ 1 + \left( u/b \right)^2 \right]^{3/2}}
\]

\[
\Delta p = \frac{eq}{2\pi c} \frac{1}{b^2} = 2
\]
The absolute value of the long impulse

\[ \Delta \rho_{\parallel} = \int_{-\infty}^{\infty} dt \, e^{\frac{\eta}{2 \gamma} \left( \gamma (z - vt) \right)} \frac{1}{\gamma c \left[ b^2 + \gamma^2 (z - vt)^2 \right]^{3/2}} \]

Taking \( v \gg c, \gamma \) large

\[ \Delta \rho_{\parallel} = e^{\frac{\eta}{2 \gamma}} \int_{-\infty}^{0} d(\gamma c t) \frac{\gamma c t}{\gamma c \left[ b^2 + (\gamma c t)^2 \right]^{3/2}} \]

\[ u = \gamma c t \]

\[ \Delta \rho_{\parallel} = \frac{e^{\frac{\eta}{2 \gamma}}}{\gamma c b} \int_{-\infty}^{0} du \frac{u/b}{\left[ 1 + (u/b)^2 \right]^{3/2}} \]

\[ = 1 \]

\[ \Delta \rho_{\parallel} = \frac{1}{\gamma c b} \frac{e^{\frac{\eta}{2 \gamma}}}{2\pi} \]
c) The electric field

\[ E_z = \frac{e}{4\pi} \frac{\gamma (z-vt)}{[b^2 + \gamma^2 (z-vt)^2]^{3/2}} \]

Since \( \gamma \to \infty \) this remains finite while becoming infinitely narrow we can discard it completely. By contrast

\[ E_z = \frac{e}{4\pi} \frac{b}{[b^2 + \gamma^2 (z-vt)^2]} \]

becomes infinitely high while becoming infinitely narrow. The previous analysis shows that even for \( \gamma \to \infty \)

\[ \int d(z-vt) E_z(b',z-vt) = \frac{e}{2\pi} \frac{b}{b^2} \]

So we have

\[ E_z = \frac{e}{2\pi} \frac{b}{b^2} S(z-ct) \]

Then the B field

\[ \overrightarrow{B} = \frac{1}{c} \times \overrightarrow{E} = \frac{e}{2\pi} \frac{\hat{V} \times \overrightarrow{b}}{b^2} S(z-ct) = \overrightarrow{B} \]
From the Maxwell eqs

\[- \partial_\alpha F^{\alpha \beta} = \frac{J^\beta}{c}\]

So

\[-\frac{\partial}{\partial x^0} F^{0 \beta} + \sum_{i=1}^{2} \frac{\partial}{\partial x^i} F^{i \beta} = \frac{J^\beta}{c}\]  \hspace{1cm} \text{we will check this for } \beta = 0, \beta = a \text{ with } a = 1, 2,

For \( \beta = 0 \):

\[-\frac{\partial}{\partial x^i} F^{i 0} = J^0\]

\[-F^{i 0} = 2E^i = \frac{e}{a} \left( \frac{b^a}{2\pi b^2} \right) \delta(2t-z)\]

Using the fact that in 2D:

\[\partial_\alpha \left( \frac{b^a}{b^2} \right) = 2\pi \delta^2(b) \]  \hspace{1cm} \text{(study the Coulomb law in 2D!)}

We see that

\[-\partial_\alpha F^{\alpha 0} = e \left( \frac{e^2}{b^2} \right) \delta(2t-z) = \frac{J^0}{c}\]
Similarly \( \beta = a = 1 \)

\[
- \frac{\partial}{\partial x^0} F^{0\alpha} + \frac{\partial}{\partial x^3} F^{3\alpha} = \frac{J^x}{c}
\]

\[
- \frac{\partial}{\partial (ct)} (E^1) + \frac{\partial}{\partial x^2} F^{21} + \frac{\partial}{\partial x^3} F^{31} = \frac{J^x}{c}
\]

From:

\[
(E^x, E^y) = e^{-i (b^x, b^y)} \delta(z - ct) \Rightarrow \mathbf{E} = F^{01}
\]

\[
\begin{align*}
(B^x, B^y) &= e^{-i (-b_y, b_x)} \delta(z - ct) \\
\mathbf{B} &= \mathbf{F}^{12} = B_x \\
\mathbf{F}^{12} &= \begin{pmatrix} 0 & B^y \\ -B^y & 0 \end{pmatrix} \\
\mathbf{F}^{13} &= -B_y = -F^{31} \\
\mathbf{F}^{31} &= \frac{e_i b_j \delta(z - ct)}{2\pi b^2}
\end{align*}
\]
So from Eq. A. becomes

\[- \frac{e}{2\pi} \frac{b^x}{b^2} \delta'(z-ct) (-1) + \frac{e}{2\pi} \frac{b^z}{b^2} \delta'(z-ct) = 0 = \frac{J^x}{c}\]

So, \( \frac{J^x}{c} = 0 \)

Similarly, \( \beta = z \)

\[- \frac{2}{\partial (c^2)} + \frac{2}{\partial x^a} F^{a\dot{c}} = \overline{J^3}\]

\[- \frac{2}{\partial x^1} F^{13} + \frac{2}{\partial x^2} F^{23} = \overline{J^3}\]

From

\[\frac{a}{2\pi} \frac{e}{b^x} \delta(z-ct) + \frac{2}{2\pi} \frac{e}{b^y} \delta(z-zt) = \frac{J^2}{c}\]

\[\frac{e}{2\pi} \left[ \frac{2}{\partial x^a (b^x/b^2)} \right] \delta(z-ct) = \frac{J^2}{c}\]

\[\frac{e}{2\pi} \delta^3(b^1) \delta(z-ct) = \frac{J^2}{c}\]
Problem 4. The Relativistic Hamiltonian

Recall that the action of a relativistic point particle is

\[ S = -mc^2 \int d\tau + \frac{e}{c} \int A_\mu(X)dX^\mu, \]  

(19)

where the trajectory \( X^\mu(\lambda) \) is parameterized by the real number \( \lambda \), and we use the short hand notation

\[ c\,d\tau \equiv \sqrt{-\frac{dX^\mu}{d\lambda} \frac{dX_\mu}{d\lambda}} \, d\lambda, \quad dX^\mu \equiv \frac{dX^\mu}{d\lambda} \, d\lambda. \]  

(20)

(a) Parameterize the path by time by taking \( \lambda = t \) and thus

\[ X^\mu(t) = (ct, \mathbf{x}(t)). \]  

(21)

Explicitly write down the action \( S[\mathbf{x}(t)] \) in terms of \( \mathbf{x}(t) \) and the familiar scalar and vector potentials \( (\varphi(t, \mathbf{x}), A(t, \mathbf{x})) \).

(b) Expand \( S[\mathbf{x}(t)] \) from part (a) for non-relativistic motion, and verify that the Euler-Lagrange equations lead to the correct non-relativistic equations of motion, with the appropriate Lorentz force \( \mathbf{F} = e(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}) \).

(c) Use the non-relativistic action of part (b) to determine the canonical momentum, \( \mathbf{p}_{\text{can}} = \partial L / \partial \dot{\mathbf{x}} \) and the Hamiltonian \( H = \mathbf{p}_{\text{can}} \cdot \dot{\mathbf{x}} - L \) in the non-relativistic limit. This is the form normally used in quantum mechanics.

(d) Now use the relativistic action \( S[\mathbf{x}(t)] \) (i.e. parameterized by time as in part (a)) to determine the canonical momentum \( \mathbf{p}_{\text{can}} \) and the relativistic Hamiltonian. You should find

\[ H = c\sqrt{(\mathbf{p}_{\text{can}} - \frac{e}{c} \mathbf{A})^2 + (mc)^2 + e \varphi(t, \mathbf{x})} \]  

(22)
The Relativistic Hamiltonian

a) \[ S = -mc^2 \int dt \sqrt{1 - \frac{x^2}{c^2}} - \int dt \left( e \phi(x,t) \right) + \int dt \frac{e \cdot \vec{A}(x,t)}{\varepsilon} \]

b) Expanding part a
\[ \sqrt{1 - \frac{v^2}{c^2}} \approx 1 - \frac{1}{2} \frac{v^2}{c^2} \]

\[ S_0 \equiv S_0 + S_{\text{int}} \]

\[ S = -mc^2 \int dt \sqrt{1 - \frac{x^2}{c^2}} + \int dt \frac{m \dot{x}^2}{2} - \int dt \left( e \phi + e \int \frac{e \cdot \vec{A}}{\varepsilon} \right) dt \]

\[ \delta S_0 = \int dt \delta x \left[ -m \ddot{x} \right] \]

\[ \delta S_{\text{int}} = \int dt \delta x \left[ -\frac{d}{dt} \frac{\partial H_{\text{int}}}{\partial \dot{x}^i} + \frac{\partial H_{\text{int}}}{\partial x^i} \right] \]

\[ = \int dt \delta x \left[ -\frac{d}{dt} \left( \frac{eA_i}{c} \right) + \frac{-e \phi}{2x^i} + \frac{e \cdot \vec{A}}{\varepsilon \varepsilon} \right] \]
Writing \( A_i = A_i(t, x(t)) \) and differentiating

\[
\frac{d}{dt}(eA_i) = \frac{d}{dt}(\text{c}) \frac{\partial A_i}{\partial t} - \frac{\partial A_i}{\partial x} \frac{\text{v} \cdot \text{e}}{c}
\]

And collecting

\[
SS = \int \text{d}t \text{d}x \left[ -m \ddot{x} + e \left( -\text{c} \frac{\partial A_i}{\partial t} - \frac{\partial A_i}{\partial x} \right) + e \left( \frac{\partial A_i}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right) \frac{\text{v} \cdot \text{d}}{\text{c}} \right]
\]

\[
SS = \int \text{d}t \text{d}x \left[ -m \ddot{x} + e E_i + e \left( \frac{\text{v} \cdot \text{B}}{\text{c}} \right)_i \right]
\]

c)

\[
P_{\text{can}} = \frac{2L}{2 \dot{x}^i}
\]

So, using part (b):

\[
P_{\text{can}} = m \dot{x}^i + e \frac{\dot{A}^i}{c}
\]

So

\[
H = P_{\text{can}} \cdot \dot{x} - L
\]

\[
= \frac{1}{m} P_{\text{can}} \cdot (\dot{P}_{\text{can}} - eA) - \left[ \left( \frac{P_{\text{can}} - eA}{c} \right)^2 - e \phi \right] + \frac{e}{c} \left( \frac{\dot{P}_{\text{can}} - eA}{m} \right) \cdot \dot{A}
\]
Where we have expressed

\[ \dot{x} = \left( \frac{\hat{P}_{\text{can}} - e\hat{A}}{c} \right)/m \]

Collecting we find

\[ H = + \left( \frac{\hat{P}_{\text{can}} - e\hat{A}}{c} \right)^2/2m + e\phi \]

Then

\[ \gamma \dot{p} = \gamma m \dot{v} + eA \]

\[ \gamma \dot{x} = \left( \frac{\hat{P}_{\text{can}} - e\hat{A}}{c} \right)/m \]

Then

\[ H = P_{\text{can}} \cdot \dot{x} - L \]

\[ = P_{\text{can}} \cdot \dot{x} - \frac{mc^2}{\gamma} + e\phi - \frac{e\gamma \dot{x} \cdot \hat{A}}{c} \]

\[ = \left( \frac{\hat{P}_{\text{can}} - eA}{c} \right) \dot{x} - \frac{mc^2}{\gamma} + e\phi \]
Then we find using $\gamma m v = (p \cos \theta - eA/c)$

$$H = mc^2 \gamma \beta^2 + mc^2 + e\phi$$

Now

$$\gamma \beta^2 + \frac{1}{\gamma} = \gamma \left( \beta^2 + \frac{1}{\gamma^2} \right) = \gamma$$

So

$$H = \gamma mc^2 + e\phi$$

Expressing

$$\gamma = \sqrt{1 + (\gamma v/c)^2}$$

and recall $\gamma m v = p - eA/c$

We find

$$H = c \left( (mc)^2 + (p - eA/c)^2 \right)^{\frac{1}{2}} + e\phi$$
Problem 5. Moving conductors

The constitutive relation is a relation between the macroscopic electrical current density in a medium and the applied fields. Recall that for a normal isotropic conductor at rest in an electric \(E\) and magnetic field \(B\) the constitutive relation in a linear response approximation is known as Ohm’s Law:

\[ J = \sigma E. \]  \hspace{1cm} (23)

The conductor is uncharged in its rest frame, but has a non-zero charge density in other frames.

(a) By making a Lorentz transformation of the current and fields for small boost velocities:

(i) Deduce the familiar constitutive relation\(^1\) for a normal conductor moving non-relativistically with velocity \(\mathbf{v}\) in an electric and magnetic field from the rest frame constitutive relation, Eq. (23). Interpret the result in terms of the Lorentz force.

(ii) Show that the charge density in the moving conductor is \(\rho \simeq \mathbf{v} \cdot \mathbf{J}/c^2\). Under what conditions is the charge density positive or negative? Does a loop of wire, which in its rest frame is uncharged and carries a current \(I\), remain uncharged when it is moving with velocity \(\mathbf{v}\)? Explain.

(b) In a general Lorentz frame the conductor moves with four velocity \(U^\mu\) (here \(U^\mu = (c, 0)\) in the conductor’s rest frame, and \(U^\mu = (\gamma c, \gamma \mathbf{v})\) in other frames). The constitutive relation in Eq. (23) can be expressed covariantly as

\[ J^\mu = \frac{\sigma}{c} F^{\mu\nu} u_\nu \]  \hspace{1cm} (24)

(i) Check that Eq. (24) reproduces the current and charge density of part (a) in the small velocity limit, \(v \ll c\).

(c) Now consider a solid conducting cylinder of radius \(R\) and conductivity \(\sigma\) rotating rather slowly with constant angular velocity \(\omega\) in a uniform magnetic field \(B_0\) perpendicular to the axis of the cylinder as shown below. Determine the current flowing in the cylinder and sketch the result.

\(^1J = \sigma(E + v/c \times B)\)
(d) Determine the torque required to maintain the cylinder’s constant angular velocity. Assume that the skin depth is much larger than the radius of the cylinder.

(e) (Optional) Evaluate the current numerically (in Amps) for a typical strong laboratory field \(\sim 1T\), and rotation frequency \(\sim 1\) Hz, for Cu wire of radius \(\sim 1\) cm.
Solution

(a) (i) In a frame where the conductor is at rest

\[ J = \sigma E \]  \hspace{1cm} (25)

the charge density \( \rho = 0 \). Make a Lorentz transformation from the conductor’s rest frame to the lab frame, i.e. a frame moving with velocity \( -u \) relative to the conductor, so that the lab observer sees the conductor moving with velocity \( u \).

We have

\[ J^\mu = \Lambda^\mu_\nu J^\nu . \]  \hspace{1cm} (26)

Here the \( J \) are the currents in the conductor frame, \( J \) are the currents in the lab frame.

To first order in \( u \) the Lorentz transformation matrix is

\[ \Lambda^\mu_\nu = \left( \begin{array}{cc} \gamma & \gamma u/c \\ \gamma u/c & \gamma \end{array} \right) \approx \left( \begin{array}{cc} 1 & u/c \\ u/c & 1 \end{array} \right) \]  \hspace{1cm} (27)

Thus

\[ J \approx u \rho + J = \sigma E \]  \hspace{1cm} (28)

We need to use the Lorentz transformation rule to relate \( \mathbf{E} \) to \( \mathbf{E} \) and \( \mathbf{B} \).

The transformation rules for the \( \mathbf{E} \) and \( \mathbf{B} \) fields are

\[ E_\parallel = E_\parallel \]  \hspace{1cm} (29)
\[ B_\parallel = B_\parallel \]  \hspace{1cm} (30)
\[ E_{\perp} = \gamma E_{\perp} - \gamma u/c \times \mathbf{B} \]  \hspace{1cm} (31)
\[ B_{\perp} = \gamma B_{\perp} + \gamma u/c \times \mathbf{E} \]  \hspace{1cm} (32)

and the inverse results

\[ E_\parallel = E_\parallel \]  \hspace{1cm} (33)
\[ B_\parallel = B_\parallel \]  \hspace{1cm} (34)
\[ E_{\perp} = \gamma E_{\perp} + \gamma u/c \times \mathbf{B} \approx E_{\perp} + u/c \times \mathbf{B} \]  \hspace{1cm} (35)
\[ B_{\perp} = \gamma B_{\perp} - \gamma u/c \times \mathbf{E} \]  \hspace{1cm} (36)

So the constitutive relation becomes to first order

\[ J = \sigma (\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}) \]  \hspace{1cm} (37)

Clearly the constitutive relation takes the form \( J = \sigma \mathbf{f} \) where \( \mathbf{f} \) is the Lorentz force.
(ii) From the lorentz transformation rules
\[
\begin{pmatrix}
\rho c \\
J^x
\end{pmatrix} \approx \begin{pmatrix}
1 & u/c \\
u/c & 1
\end{pmatrix} \begin{pmatrix}
0 \\
J^x
\end{pmatrix}
\]
(38)

So the charge density is
\[
\rho c \approx uJ^x/c \approx uJ^x/c 
\]
where we used again that \( J \approx J \) to first order in \( v/c \). Or, using vectors we have
\[
\rho = u \cdot J/c^2
\]
(40)

and note that \( u \approx v \) in the non-relativistic limit. From this expression we see that a moving loop of wire carrying current \( I \) has positive charge density when the current is in the same direction as the direction of motion of the conductor. But, the charge density is negative when the current is in the opposite direction of the direction of motion of the conductor. For a closed loop of wire the total charge is unchanged and is equal to zero.

(b) We have with \( U^\mu = (\gamma c, \gamma v) \)
\[
J^i = \frac{\sigma}{c} F^{i\mu} U_\mu 
\]
(41)
\[
= \frac{\sigma}{c} [E^i + (\gamma v \times B)^i] 
\]
(42)
\[
\approx \sigma [E^i + (v/c \times B)^i] 
\]
(43)

(c) Using the result
\[
J = \sigma (u/c \times B_o),
\]
(44)
we find in cylindrical coordinates
\[
J(\rho, \phi) = -\sigma \frac{\omega \rho B_o}{c} \cos \phi \hat{z}.
\]
(45)

We see that the electrons (which carry negative charge) flow up the wire at \( \phi = 0 \) and down the wire at \( \phi = \pi \).

(d) Then Lorentz force on the current induces a torque:
\[
\tau = \int d^3 r \ r \times \left( \frac{J}{c} \times B_o \right),
\]
(46)
\[
= L \int \rho d\rho d\phi \left[ \frac{J}{c} (r \cdot B_o) - (r \cdot J/c) B_o \right],
\]
(47)

where \( L \) is the length of the cylinder. The second term in square braces integrates to zero while the first terms gives
\[
\tau = L \int_0^R \rho d\rho \int d\phi \left[ -\sigma \frac{\omega \rho B_o}{c^2} \cos \phi \hat{z} (\rho \cos \phi B_o) \right],
\]
(48)
\[
= -L \hat{z} \frac{\pi \sigma \omega R^4 B_o^3}{4c^2}.
\]
(49)
This is the torque by the magnetic field on the cylinder. To maintain a constant angular velocity we need an external torque per length of

\[ \frac{\tau}{L} = \pm z \frac{\pi \sigma \omega R^4 B_o^2}{4e^2}. \]  

**(50)**

**Notes:**

- An alternative way to derive this is to equate the work done per time by the external torque, \( \tau \cdot \omega \), with the energy dissipation

\[ \tau \cdot \omega = \int d^3r \frac{J \cdot J}{\sigma} \]  

\[ = L \frac{\sigma \omega^2}{4c^2} B_o^2 \pi R^4 \]  

**(52)**

- We next evaluate this numerically for copper. Expressing the torque in terms of the skin depth (which is taken from Wikipedia):

\[ \delta = \sqrt{\frac{2e^2}{\sigma \omega}} = 6.5 \text{ cm} / \sqrt{f_{Hz}} \]  

**(53)**

We find

\[ \frac{\tau}{L} = \frac{R^4 B_o^2 \pi}{\delta^2 / 2} \]  

**(54)**

Converting to MKS and Tesla

\[ B_o^2 \rightarrow \frac{B_o^2}{\mu_o} = 1 \frac{J}{m^2} 8 \times 10^5 \left( \frac{B_o}{\text{Tesla}} \right)^2 \]  

**(55)**

So we find

\[ \frac{\tau}{L} \approx 3 N \left( \frac{R}{\text{cm}} \right)^4 \left( \frac{f}{\text{Hz}} \right) \left( \frac{B_o}{\text{Tesla}} \right)^2 \quad \text{with} \quad R \ll \frac{6.5 \text{ cm}}{\sqrt{f \text{ in Hz}}} \]  

**(56)**

It is also to calculate the current flowing through each hemi-cylinder of the wire.

\[ \frac{I}{c} = \int \rho d\rho \int_{-\pi/2}^{\pi/2} d\phi J(\rho, \phi)/c \]  

\[ = - \frac{2}{3} \frac{\omega R^3 B_o}{c^2} \hat{z} \]  

**(58)**

\[ = - \frac{4 R^3 B_o}{3 \delta^2} \]  

**(59)**

Or in MKS

\[ \frac{I}{c} \rightarrow \sqrt{\mu_o} I \]  

**(60)**

\[ B \rightarrow \frac{B}{\sqrt{\mu_o}} \]  

**(61)**
which evaluates to a shockingly large current

\[ I = -\frac{4 R^3 B_o}{3 \delta^2 \mu_o} \]  
\[ = 251 \text{Amps} \left( \frac{f}{\text{Hz}} \right) \left( \frac{B}{\text{Tesla}} \right) \left( \frac{R}{\text{cm}} \right)^3 \]