Problem 1. Lienard-Wiechert for constant velocity

(a) For a particle moving with constant velocity v along the $x-$axis show using Lorentz transformation that gauge potential from a point particle is

$$A^x(t, x, \boldsymbol{x}_\perp = \boldsymbol{b}) = \frac{e}{4\pi} \frac{\gamma\beta}{\sqrt{b^2 + \gamma^2(x - vt)^2}} \tag{1}$$

at the point $(t, \boldsymbol{r}) = (t, x, y, z) = (t, x, \boldsymbol{b})$. So at the point $(t, 0, b, 0)$ the gauge potential A^x is is

$$A^x(t, x, y = b) = \frac{e}{4\pi} \frac{\gamma\beta}{\sqrt{b^2 + (\gamma vt)^2}} \tag{2}$$

(b) Start by noting the definitions

$$T \equiv t - \frac{R}{c} \qquad R = |\boldsymbol{r} - \boldsymbol{r}_*(T)| \qquad \boldsymbol{R} \equiv R\boldsymbol{n} \equiv \boldsymbol{r} - \boldsymbol{r}_*(T) \qquad \boldsymbol{n} \equiv \frac{\boldsymbol{R}}{R} \tag{3}$$

and drawing a picture for yourself. Then, after setting $c = 1$ and $v = \beta$ to simplify algebra, show that the Lienard Wiechert result,

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{e}{4\pi} \left[\frac{\boldsymbol{v}/c}{R(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})} \right]_{\rm ret} . \tag{4}$$

gives the same result as Eq. (2).

(c) Show that the Lienard-Wiechert potential, Eq. (4), and analogous equation for Φ can be written covariantly

$$A^\mu(X) = -\frac{e}{4\pi} \left[\frac{U^\mu}{U \cdot \Delta X} \right]_{\rm ret} , \tag{5}$$

where ΔX^μ is the difference in the space-time coordinate four vectors of the emission and observation points, and U^μ is the four velocity of the particle. What is $\Delta X \cdot \Delta X \equiv \Delta X^\mu \Delta X_\mu$? Can $[]_{\rm ret}$ be expressed covariantly?

Problem 2. The current and a tutorial on variational derivatives

- Variational derivatives cause students great hardship. Its meaning is discussed in what follows. We are considering an integral[1] depending on a path $x(t)$ starting at x_1 and ending at x_2. For example

$$I[x] = \int_{t_1,x_1}^{t_2,x_2} dt\, L(x(t))\,. \tag{6}$$

Then we deform the path

$$x(t) \to x(t) + \delta x(t) \tag{7}$$

where the endpoints are unchanged $\delta x(t_1) = \delta x(t_2) = 0$. Then the integral changes and the result must be proportional to $\delta x(t)$ for smal variations

$$\delta I[x] = \int dt \left[\frac{\partial L(x(t))}{\partial x(t)} \right] \delta x(t) \tag{8}$$

We say that the thing in square bracekts (i.e. the thing sitting in front of $\int dt\, \delta x(t)$) is the variation derivative of the functional

$$\frac{\delta I[x]}{\delta x(t)} = \text{thing in front of } \textstyle\int dt\, \delta x(t) \quad = \frac{\partial L(x(t))}{\partial x(t)} \tag{9}$$

When working with variations, I prefer to work with the change in the integral (i.e. Eq. (8)), which somehow means more to me than some mysterious new differentiation symbol, and always works.

- However, as the formalism of variational derivatives is common, let us develop it. Clearly

$$x(t) = \int dt\, x(t')\, \delta(t - t')\,. \tag{10}$$

Then following the steps leading to Eq. (8) and Eq. (9) we see that

$$\frac{\delta x(t)}{\delta x(t')} = \delta(t - t')\,. \tag{11}$$

Then the normal rules of differentiation apply

$$\frac{\delta L(x(t'))}{\delta x(t)} \equiv \frac{\partial L(x(t'))}{\partial x(t')} \frac{\delta x(t')}{\delta x(t)} = \frac{\partial L(x(t'))}{\partial x(t')} \delta(t' - t)\,. \tag{12}$$

In this way if

$$I[x] = \int_{t_1,x_1}^{t_2,x_2} dt'\, L(x(t'))\,, \tag{13}$$

[1] Technically the integral is a functional of $x(t)$, *i.e.* something which takes a function ($x(t)$) and spits out a number.

then we can differentiate under the integral

$$\frac{\delta I[x]}{\delta x(t)} = \int_{t_1,x_1}^{t_2,x_2} dt' \, \frac{\delta L(x(t'))}{\delta x(t)} \,, \tag{14}$$

$$= \int_{t_1,x_1}^{t_2,x_2} dt' \, \frac{\partial L(x(t'))}{\partial x(t)} \delta(t'-t) \tag{15}$$

$$= \frac{\partial L(x(t))}{\partial x(t)} \,, \tag{16}$$

as we got before

- Some people who do numerics like to work discretely where $x_i = x(t_i)$, with $t_i = t_1 + (i-1)\Delta t$ being discretely spaced points, $x_0, x_1, \ldots$. Then the integral is an ordinary function of x_i

$$I(x_1, x_2, x_3 \ldots) = \sum_i \Delta t L(x_i) \tag{17}$$

  Then the variational derivative is just limit as Δt goes to zero of

$$\frac{\delta I[x]}{\delta x(t_i)} = \frac{1}{\Delta t} \frac{\partial I}{\partial x_i} \tag{18}$$

- We have discussed a function of t and the integral which is a functional of $x(t)$. When working with fields which are a function of space-time $A(x)$ (here $x = (ct, \boldsymbol{x})$), the integral is functional of $A(x)$

$$I[A] = \int d^4x \, \mathcal{L}(A(x)) \,. \tag{19}$$

  Then the variation of the integral is found by changing the function $A(x)$ to a new function

$$A(x) \to A(x) + \delta A(x) \,. \tag{20}$$

  The integral then changes to $I \to I + \delta I$

$$\delta I = \int d^4x \left[\frac{\partial \mathcal{L}(A(x))}{\partial A(x)} \right] \delta A(x) \tag{21}$$

  The thing in square brackets in front of $\int d^4x \, \delta A(x)$ is defined as the variational derivative

$$\frac{\delta I[A]}{\delta A(x)} = \text{thing in front of } \int d^4x \, \delta A(x) \tag{22}$$

$$= \frac{\partial \mathcal{L}(A(x))}{\partial A(x)} \text{in this simple case} \tag{23}$$

- In the same sense as before

$$A(x) = \int d^4y A(y) \delta^4(x-y) \,. \tag{24}$$

Thus

$$\frac{\delta A(x)}{\delta A(y)} = \delta^4(x-y)\,, \tag{25}$$

and

$$\frac{\delta \mathcal{L}(A(y))}{\delta A(x)} \equiv \frac{\partial \mathcal{L}(A(y))}{\partial A(y)} \delta^4(y-x)\,. \tag{26}$$

I have always found this slightly confusing and a bit too formal, and prefer the more understandable change in integral, Eq. (21).

We defined the current as the variation of the action describing the interaction between the particles and the gauge field A^μ, i.e. the thing in front of $\int d^4x \delta A_\mu \ldots$

$$\delta S_{\rm int}[A] = \int d^4x\, (J^\mu/c)\, \delta A_\mu \tag{27}$$

This is often written

$$\frac{J^\mu}{c} = \frac{\delta S}{\delta A_\mu(t,\boldsymbol{x})} \tag{28}$$

but means the same as Eq. (27)

(a) Consider the action of a non-relativistic point particle, $\boldsymbol{x}_o(t)$

$$S = \int dt \frac{1}{2} m\, \dot{\boldsymbol{x}}_o^2 + \frac{e}{c}\int dt\; \boldsymbol{v}_o(t)\cdot \boldsymbol{A}(\boldsymbol{x}_o(t)) - \int dt\; e\, \Phi(t, \boldsymbol{x}_o) \tag{29}$$

Compute the current and charge density by varying the action. You should find an "obvious" result. Hint: first write

$$\boldsymbol{A}(t, \boldsymbol{x}_o(t)) = \int d^3\boldsymbol{x}\; \boldsymbol{A}(t,\boldsymbol{x})\, \delta^3(\boldsymbol{x} - \boldsymbol{x}_o(t)) \tag{30}$$

(b) Compute J^μ for a relativistic particle $X_o^\mu(\tau)$

$$S = -mc^2 \int d\tau + \frac{e}{c}\int A_\mu(X_o) dX_o^\mu \tag{31}$$

(c) Show that if $S_{\rm int}[A]$ is gauge invariant that the current defined by Eq. (27) is automatically conserved.

Problem 3. (Optional but please read) The current in non-relatitivistic quantum mechanics

Consider the Lagrangian of a non-relativistic quantum mechanical particle. We need a Lagrangian for the wave function $\psi(x)$. We set $\hbar = 1$.

The action of a non-relatitivistic particle is

$$S[\psi, \psi^*] = \int dt d^3x \ \psi^* i\partial_t \psi - \mathcal{H} \tag{32}$$

Here

$$\mathcal{H} = \psi^* \frac{-\nabla^2}{2m} \psi + e\psi^*(x)\psi(x)\, \Phi(x) \tag{33}$$

One can also integrate by parts and write this as

$$S[\psi, \psi^*] = \int dt d^3x \ \psi^* i\partial_t \psi - \frac{(-i\partial_i \psi)^*(-i\partial^i \psi)}{2m} - e\Phi(t,x)\psi^*(t,x)\psi(t,x) \tag{34}$$

(a) By varying the action with respect to ψ and ψ^* (treat them as independent variables) determine the equation of motion

$$\frac{\delta S[\psi, \psi^*]}{\delta \psi(t, \boldsymbol{x})} = 0 \qquad \frac{\delta S[\psi, \psi^*]}{\delta \psi^*(t, \boldsymbol{x})} = 0 \tag{35}$$

Comment on the result

(b) In the presence of a vector potential $A_i(t, \boldsymbol{x})$ we $\boldsymbol{p} \to \boldsymbol{p} - e\boldsymbol{A}/c$. Then the Lagrangian is then the same with the replacement

$$-i\partial_i \to -i\partial_i - \frac{e}{c} A_i(x) \tag{36}$$

Make this replacement. Determine the current at vanishing external field by varrying the action and setting $A_i = 0$

$$\frac{J^i}{c} = \left. \frac{\delta S}{\delta A_i} \right|_{A_i = 0} \tag{37}$$

You should find the ususal quantum mechanical probability current

$$\boldsymbol{J} = \frac{1}{2m} \left[\psi^*(-i\boldsymbol{\nabla}\psi) + (-i\boldsymbol{\nabla}\psi)^*\psi \right] \tag{38}$$

Problem 4. Energy during a burst of deceleration

A particle of charge e moves at constant velocity, βc, for $t < 0$. During the short time interval, $0 < t < \Delta t$ its velocity remains in the same direction but its speed decreases linearly in time to zero. For $t > \Delta t$, the particle remains at rest.

(a) Show that the radiant energy emitted per unit solid angle is

$$\frac{dW}{d\Omega} = \frac{e^2\beta^2}{64\pi^2 c\Delta t} \frac{(2-\beta\cos\theta)\left[1+(1-\beta\cos\theta)^2\right]\sin^2\theta}{(1-\beta\cos\theta)^4} \tag{39}$$

(b) In the limit $\gamma \gg 1$, show that the angular distribution can be expressed as

$$\frac{dW}{d\xi} \simeq \frac{e^2\beta^2}{4\pi\, c} \frac{\gamma^4}{\Delta t} \frac{\xi}{(1+\xi)^4} \tag{40}$$

where $\xi = (\gamma\theta)^2$.

(c) Show for $\gamma \gg 1$ that the total energy radiated is in agreement with the relativistic generalization of the Larmour formula.

Problem 5. An oscillator radiating

(a) Determine the time averaged power radiated per unit sold angle for a *non-relativistic charge* moving along the z-axis with instantaneous position, $z(T) = H\cos(\omega_o T)$.

(b) Now consider relativistic charge executing simple harmonic motion. Show that the instantaneous power radiated per unit solid angle is

$$\frac{dP(T)}{d\Omega} = \frac{dW}{dT\, d\Omega} = \frac{e^2}{16\pi^2}\frac{c\beta^4}{H^2}\frac{\sin^2\theta\cos^2(\omega_o T)}{(1+\beta\cos\Theta\sin\omega_o T)^5} \tag{41}$$

Here $\beta = \omega_o H/c$ and $\gamma = 1/\sqrt{1-\beta^2}$

(c) In the relativistic limit the power radiated is dominated by the energy radiated during a short time interval around $\omega_o T = \pi/2,\ 3\pi/2,\ 5\pi/2, \ldots$. Explain why. Where does the outgoing radiation point at these times.

(d) Let ΔT denote the time deviation from one of this discrete times, *e.g.* $T = 3\pi/(2\omega_o) + \Delta T$. Show that close to one of these time moments:

$$\frac{dP(\Delta T)}{d\Omega} = \frac{dW}{d\Delta T\, d\Omega} \simeq \frac{2e^2}{\pi^2}\frac{c\beta^4}{H^2}\gamma^6\frac{(\gamma\omega_o\Delta T)^2(\gamma\theta)^2}{(1+(\gamma\theta)^2+(\gamma\omega_o\Delta T)^2)^5} \tag{42}$$

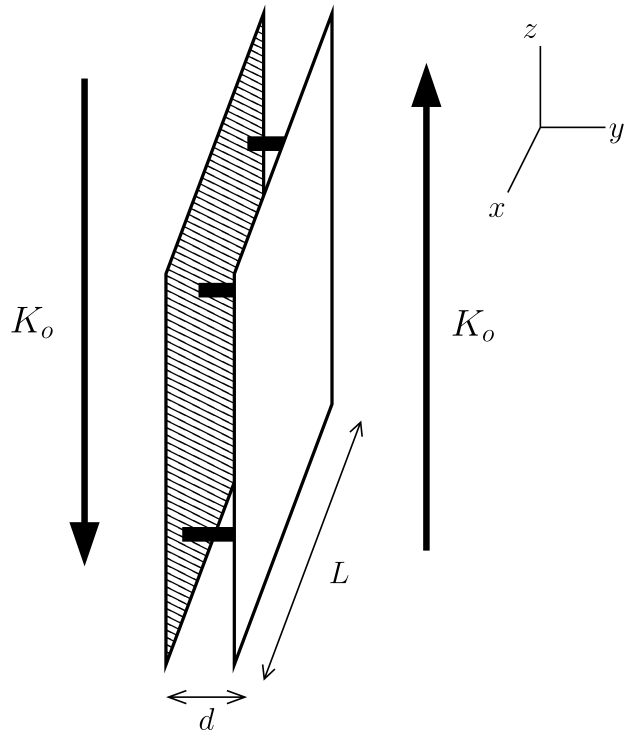
(e) By integrating the results of the previous part over the ΔT of a single pulse, show that the time averaged power is

$$\overline{\frac{dP(T)}{d\Omega}} = \frac{e^2}{128\pi^2}\frac{c\beta^4}{H^2}\gamma^5\frac{5(\gamma\theta)^2}{(1+(\gamma\theta)^2)^{7/2}} \tag{43}$$

(f) Make rough sketches of the angular distribution for non-relativistic and relativistic motion.

Problem 6. Two current sheets under Lorentz boosts

Consider two large square sheets of conducting material (with sides of length L separated by a distance d, $d \ll L$) each carrying a uniform surface current of magnitude K_o. (The total current in each sheet is $I_o = K_o L$.) The current flows up the right sheet and returns down the left sheet. The mass of the sheets is negligible. The sheets are mechanically supported by four electrically neutral columns of mass $M_{\rm col}$ and cross sectional area $A_{\rm col}$ (three shown). Neglect all fringing fields.



(a) (3 points) Write down the electromagnetic stress tensor $\Theta^{\mu\nu}_{\rm em}$ covariantly in terms of $F^{\mu\nu}$ and compute all non-vanishing components of $F^{\mu\nu}$ and $\Theta^{\mu\nu}_{\rm em}$ inside and outside of the sheets.

(b) (1 point) Compute the total rest energy of the system (or $M_{\rm tot}c^2$) including the contribution from the electromagnetic energy.

(c) (3 points) Determine the electromagnetic force per area on the current sheets (magnitude and direction) and the components of the mechanical stress tensor in the columns, $\Theta^{00}_{\rm mech}$ and $\Theta^{yy}_{\rm mech}$ (use the coordinates system in the figure). You can assume that the stress is constant across the cross sectional area of the columns.

(d) (6 points) Now consider the system according to an observer moving relativistically with velocity $\beta = v/c$ up the z-axis.

   (i) Determine the electric and magnetic fields (magnitudes and directions) using a Lorentz transformation. Check that direction of the Poynting vector measured by this observer is consistent with physical intuition.

(ii) Determine the charge and current densities in the sheets according to this observer. Are your charges and currents consistent with the fields computed in the first part of (d)? Explain.

(e) (7 points) Now consider the system according to an observer moving relativistically with velocity $\beta = v/z$ to the *right* along the y-axis (use the coordinate system shown in the figure).

(i) Determine the total mechanical energy in the columns according to this observer.

(ii) Determine the total electromagnetic energy according to this observer.

(iii) Determine the total energy of this configuration. Are your results consistent with part (b)? Explain.

Comment: There is of course stress in the sheets. But, since it does not have a yy component the stress in the sheets can be neglected in this problem.