## Problem 1. Periodic pulses

Consider a periodic motion that repeats itself with period $\mathcal{T}_o$. Show that the continuous frequency spectrum becomes a discrete spectrum containing frequencies that are integral multiples of the fundamental, $\omega_o = 2\pi/\mathcal{T}_o$.

Let the electric field from a single pulse (or period) be $E_1(t)$, *i.e.* where $E_1(t)$ is non-zero between 0 and $\mathcal{T}_o$ and vanishes elsewhere, $t < 0$ and $t > \mathcal{T}_o$. Let $E_1(\omega)$ be its fourier transform.

(a) Suppose that the wave form repeats once so that two pulses are received. $E_2(t)$ consists of the first pulse $E_1(t)$, plus a second pulse, $E_2(t) = E_1(t) + E_1(t - \mathcal{T}_o)$. Show that the Fourier transform and the power spectrum is

$$E_2(\omega) = E_1(\omega)\,(1 + e^{i\omega\mathcal{T}_o}) \qquad |E_2(\omega)|^2 = |E_1(\omega)|^2\,(2 + 2\cos(\omega\mathcal{T}_o)) \tag{1}$$

(b) Now suppose that we have $n$ (with $n$ odd) arranged almost symmetrically around $t = 0$, *i.e.*

$$E_n(t) = E_1(t+(n-1)\mathcal{T}_o/2)+\ldots+E_1(t+\mathcal{T}_o)+E_1(t)+E_1(t-\mathcal{T}_o)+\ldots E_1(t-(n-1)\mathcal{T}_o/2)\,, \tag{2}$$

so that for $n = 3$

$$E_3(t) = E_1(t + \mathcal{T}_o) + E_1(t) + E_1(t - \mathcal{T}_o)\,. \tag{3}$$

Show that

$$E_n(\omega) = E_1(\omega)\,\frac{\sin(n\omega\mathcal{T}_o/2)}{\sin(\omega\mathcal{T}_o/2)} \tag{4}$$

and

$$|E_n(\omega)|^2 = |E_1(\omega)|^2\left(\frac{\sin(n\omega\mathcal{T}_o/2)}{\sin(\omega\mathcal{T}_o/2)}\right)^2 \tag{5}$$

(c) By taking limits of your expressions in the previous part show that after $n$ pulses, with $n \to \infty$, we find

$$E_n(\omega) = \sum_m E_1(\omega_m)\frac{2\pi}{\mathcal{T}_o}\delta(\omega - \omega_m) \tag{6}$$

and

$$|E_n(\omega)|^2 = \underbrace{n\mathcal{T}_o}_{\text{total time}} \times \sum_m |E_1(\omega_m)|^2\frac{2\pi}{\mathcal{T}_o^2}\delta(\omega - \omega_m) \tag{7}$$

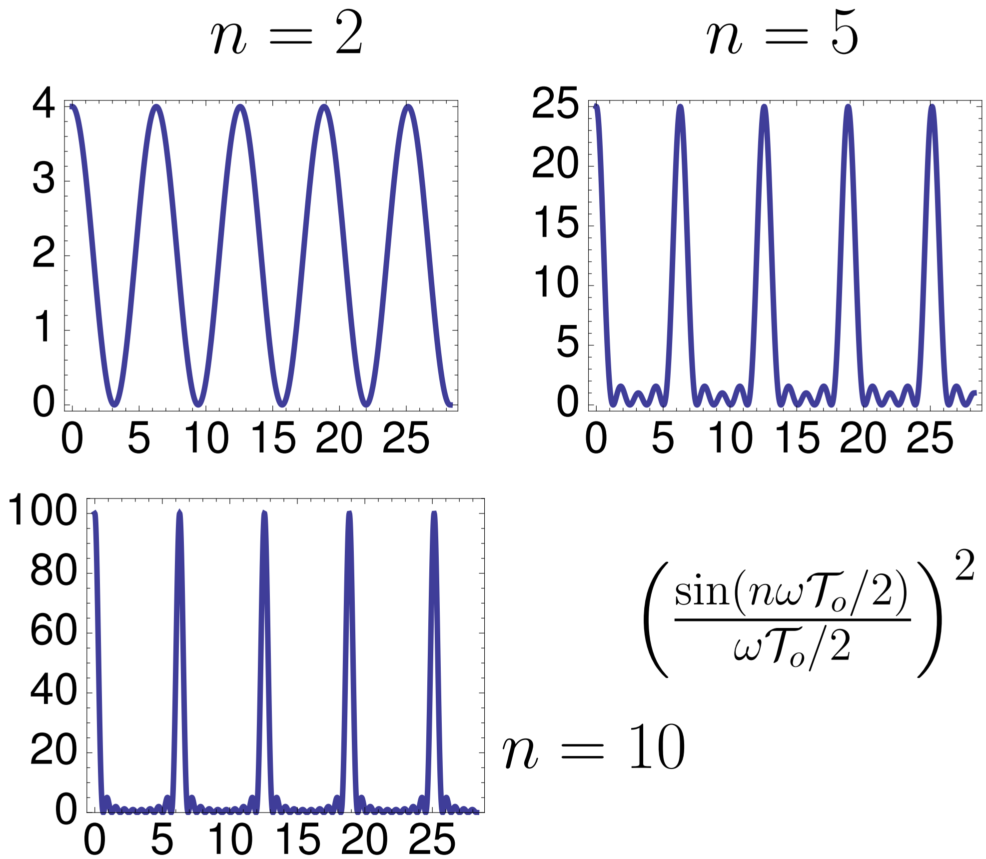
where $\omega_m = 2\pi m/\mathcal{T}_o$.

**Remark** We have in effect shown that if we define

$$\Delta(t) \equiv \sum_{n=-\infty}^{\infty} \delta(t - n\mathcal{T}_o)\,. \tag{8}$$

Then the Fourier transform of $\Delta(t)$ is

$$\hat{\Delta}(\omega) = \sum_n e^{-i\omega n\mathcal{T}_o} = \sum_m \frac{2\pi}{\mathcal{T}_o}\delta(\omega - \omega_m)\,. \tag{9}$$



n = 2
n = 5
n = 10
4
3
2
1
0
25
20
15
10
5
100
80
60
40
0 5 10 15 20 25
$\left(\frac{\sin(n\omega\mathcal{T}_o/2)}{\omega\mathcal{T}_o/2}\right)^2$

(d) Show that a general expression for the time averaged power radiated per unit solid angle into each multipole $\omega_m \equiv m\omega_o$ is:

$$\frac{dP_m}{d\Omega} = \frac{|rE(\omega_m)|^2}{\mathcal{T}_o^2} \tag{10}$$

Or

$$\frac{d\hat{P}_m}{d\Omega} = \frac{e^2\omega_o^4 m^2}{32\pi^4 c^3}\left|\int_0^{\mathcal{T}_o} \mathbf{v}(T)\times\boldsymbol{n}\exp\left[i\omega_m(T - \frac{\boldsymbol{n}\cdot r_*(T)}{c})\right]\right|^2 \mathrm{d}T\,, \tag{11}$$

Here $d\hat{P}_m/d\Omega$ is defined so that over along time period $\Delta\mathcal{T}$, the energy per solid angle is

$$\frac{dW}{d\Omega} = \Delta\mathcal{T}\sum_{m=1}^{\infty}\frac{d\hat{P}_m}{d\Omega} \tag{12}$$

Also note that we are summing only over the positive values of $m$ which is different from how we had it in class:

$$\frac{d\hat{P}_m}{d\Omega} \equiv \frac{dP_m}{d\Omega} + \frac{dP_{-m}}{d\Omega} \tag{13}$$

## Problem 2. Radiation spectrum of a SHO

(a) Show that for the simple harmonic motion of a charge discussed in Problem 2 the average power radiated per unit solid angle in the $m$-th harmonic is

$$\frac{d\hat{P}_m}{d\Omega} = \frac{e^2 c\beta^2}{8\pi^2 H^2} m^2 \tan^2\theta \left[J_m(m\beta\cos\theta)\right]^2 \tag{14}$$

(b) Show that in the non-relativistic limit the total power radiated is all in the fundamental and has the value

$$P = \frac{e^2}{4\pi}\frac{2}{3}\omega_o^4 \overline{H^2} \tag{15}$$

where $\overline{H^2}$ is the mean squared amplitude of the oscillation.

## Problem 3. Radiation spectrum from a damped SHO

The non-relativistic motion of a charged particle of charge $e$ is described by a damped harmonic oscillator

$$m\frac{d^2z}{dt^2} + m\eta\frac{dz}{dt} + m\omega_o^2 z = 0 \tag{16}$$

where $\eta$ is small, $\eta \ll \omega_o$. Also assume that $\Delta\omega \equiv \omega - \omega_o \ll \omega_o$. Be sure to use these approximations at all points of the clculation.

The charge is released from rest with initial amplitude $z(t=0) = H$.

(a) On the $x$ axis, far from the charge, how is the light polarized ?

(b) Estimate (i.e. don't calculate) the energy lost per time to radiation. We will require that the energy lost to radation is small compared to energy lost to friction. How does this requirement constrain the dimensionful parameters of this problem: $m, H, \omega_o, \eta, e, c$

(c) Determine the spectrum of photons which are emitted

$$\omega\frac{dN}{d\omega} = \frac{1}{\hbar}\frac{dI}{d\omega} = \frac{2}{\hbar}\left.\frac{dW}{d\omega}\right|_{\omega>0} \tag{17}$$

(The factor of two incorporates the contributions with $\omega < 0$, which give an equal contribution. Why?) Express your final result in terms of the fine structure constant $\alpha$ instead of the charge (squared).

(d) **Optional – but extremely good practice for exam** Integrate the results of the previous part over frequency to determine the total energy that is emitted. Calculate the same result by integrating the Larmour formula

$$P(t_e) = \frac{q^2}{4\pi}\frac{2}{3}\frac{a^2(t_e)}{c^3} \tag{18}$$

over time.

(e) **Optional** In part (c) you determine the frequency spectrum for $\Delta\omega \ll \omega_o$. In part (d) you integrated over $\Delta\omega$ (from $-\infty \ldots \infty$) to determine the total power. Estimate the error made by extending this integral over the full frequency range instead of just a narrow range around $\omega_o$. Similarly estimate the error in your approximate formula for the acceleration.

## Problem 4. Soft bremsstrahlung during a decay

In a collision or decay that happens at location $\boldsymbol{r}_o$ over an infinitessimally short time scale, $\tau_{\rm accel}$, the charged particles moving with velocity, $\mathbf{v}_1, \mathbf{v}_2, \ldots$ before the collsions and the charged particles moving with $\mathbf{v}_{1'}, \mathbf{v}_{2'}, \ldots$, after the collision each contribute to the radiation field. (The total radiation field is just a sum of the radiation fields from each particle.)

(a) Show that for frequencies low $\omega \ll 1/\tau_{\rm accel}$ the total radiation field is

$$\boldsymbol{E}_{\rm rad}(\omega, r) = e^{i\omega(r-\boldsymbol{n}\cdot\boldsymbol{r}_o)/c}\left(\sum_{j' \,\in\, \rm final} \frac{q_{j'}}{4\pi r c^2}\frac{\boldsymbol{n}\times\boldsymbol{n}\times\mathbf{v}_{j'}}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}_{j'}} - \sum_{j \,\in\, \rm initial} \frac{q_j}{4\pi r c^2}\frac{\boldsymbol{n}\times\boldsymbol{n}\times\mathbf{v}_j}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}_j}\right) \tag{19}$$

This generalizes the result of Lecture 46.

Hint. You may encounter an integral like

$$\int_0^\infty \boldsymbol{n}\times\boldsymbol{n}\times\mathbf{v}\; e^{i\omega T(1-\boldsymbol{n}\cdot\mathbf{v}/c)} \;. \tag{20}$$

To give this integral definite meaning insert a convergence factor $e^{-\epsilon|T|}$ and then take the limit $\epsilon \to 0$ after integration. In any real experiment the velocity $\mathbf{v}(T)$ would be cut off in time, and provide this convergence factor naturally.

(b) A neutral $\omega^o$ meson of mass $M_\omega c^2 = 784\,\text{MeV}$ has a relatively rare decay mode $\omega^o \to \pi^+\pi^-$, with branching fraction of 1.53%. (98.5% of the time it decays to something else.) It has another rare decay mode $\omega^o \to e^+e^-$ with branching ratio $7.28\times10^{-3}$%. (These are pretty rare decays for the $\omega^o$ meson – most of the time it decays to $\pi^+\pi^-\pi^0$ with a branching fraction of 89.2%). The mass of a pion is $mc^2 = 140\,\text{MeV}$, while the electron mass is ...

(i) Compute the frequency spectrum of the soft electromagnetic radiation per solid angle that accompanies both of these decay modes

$$\frac{dI}{d\omega d\Omega} = 2\left.\frac{dW}{d\omega d\Omega}\right|_{\omega>0}\,, \tag{21}$$

Describe your result qualitatively.

(ii) Show that for both of these decay modes the frequency spectrum of radiated energy at low frequencies is

$$\frac{dI}{d\omega} = \frac{e^2}{4\pi^2 c}\left[\left(\frac{1+\beta^2}{\beta}\right)\ln\frac{1+\beta}{1-\beta} - 2\right] \simeq \frac{e^2}{\pi^2 c}\left[\ln\left(\frac{M_\omega}{m}\right) - \frac{1}{2}\right] \tag{22}$$

where $M_\omega$ is the mass of the $\omega_o$ meson, $m$ is the mass of one of the decay products, and $\beta$ is the velocity/c of the decay products.

(iii) Roughly evaluate the total energy radiated in each decay by integrating the spectrum up to a point where the photon's momentum is half of the momentum of the decay products. (Beyond this point the recoil of the charged decay products would need to be considered. This lies outside of classical electrodynamics. In classical electrodynamics we specify the currents and solve for the fields.). You should find in a leading $\log(M_\omega/m)$ approximation

$$\frac{I_{\rm rough}}{M_\omega c^2} \simeq \frac{\alpha}{\pi} \log\left(\frac{M_\omega}{m}\right) \tag{23}$$

Using this rough evaluation, what fraction of the rest energy of the $\omega^o$ is carried away by soft radiation in the two decay modes

## Problem 5. Thomson Scattering

We will do this in class. It is very important, especially for astrophysics.

(i) Polarized light with linear polarization vector $\boldsymbol{\epsilon}_o$, is propagating in the $z$-direction with electric field amplitude $E_o$ and is incident upon an electron at rest. Assume that $\hbar\omega$ is much less than the electron mass $m_e c^2$. Show that the time average power radiated into light with polarization $\boldsymbol{\epsilon}$ is

$$\left\langle \frac{dP_{\rm pol}}{d\Omega} \right\rangle = \tfrac{1}{2} c E_o^2 \left( \frac{e^2}{4\pi\, m_e c^2} \right)^2 |\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_o|^2 \tag{24}$$

where $\boldsymbol{\epsilon}$ is the polarization of the outgoing radiation, *i.e.* $\boldsymbol{n} \cdot \boldsymbol{\epsilon} = \boldsymbol{z} \cdot \boldsymbol{\epsilon}_o = 0$.

(ii) Show that the time averaged power radiated into light of any polarization by an incident beam with polarization $\epsilon_o$ is

$$\left\langle \frac{dP_{\rm unpol}}{d\Omega} \right\rangle = \tfrac{1}{2} c E_o^2 \left( \frac{e^2}{4\pi\, m_e c^2} \right)^2 |\boldsymbol{n} \times \boldsymbol{\epsilon}_o|^2 \tag{25}$$

(iii) Show that the polarized and unpolarized cross sections for incident light with polarization $\boldsymbol{\epsilon}_o$ are

$$\frac{d\sigma_{\rm pol}}{d\Omega} = r_e^2 \, |\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_o|^2 \tag{26}$$

and

$$\frac{d\sigma_{\rm unpol}}{d\Omega} = r_e^2 \, |\boldsymbol{n} \times \boldsymbol{\epsilon}_o|^2 \ , \tag{27}$$

respectively. Here the classical electromagnetic radius is

$$r_e = \frac{e^2}{(4\pi) m_e c^2} \tag{28}$$

(iv) By sticking in appropriate powers of $\hbar$, show that $r_e$ is 137 times smaller than the compton wavelength, $\lambda\!\!\!^{-}_C = \hbar/m_e c$. Show that $r_e$ is $(137)^2$ times smaller than the Bohr radius.

**Remark:** A heuristic way to understand why $r_e$ is smaller than the "the size of an electron" , $\hbar/m_e c$, is that the cross section is the cross-sectional area $\propto (\hbar/m_e c)^2$ of the electron times the probability that the light will actually interact with the electron, wich is $\alpha^2$.

(v) Now consider unpolarized incident light (light which is equally likely to be polarized in the $x$ or $y$ directions). Let the radiation be scattered at an angle $\theta$ in the $xz$ plane, where $\boldsymbol{n} \cdot \boldsymbol{n}_o = \cos\theta$. Depending on the scattering angle $\theta$, the outgoing light will be partially polarized in the $xz$ plane, or out of the $xz$ plane (*i.e.* in the $y$ direction).

Show that the cross-section for unpolarized light to produce in-plane polarized light is

$$\frac{d\sigma_{\parallel}}{d\Omega} = \tfrac{1}{2} r_e^2 \cos^2\theta \tag{29}$$

while the cross-section to produce out-of-plane polarized light is

$$\frac{d\sigma_\perp}{d\Omega} = \tfrac{1}{2} r_e^2 \tag{30}$$

And conclude that the cross-section for unpolarized light to produce light of any polarization is

$$\frac{d\sigma}{d\Omega} = r_e^2 \frac{1+\cos^2\theta}{2} \tag{31}$$

(vi) By using the results of this problem and integrating over angles, or appealing directly to the Larmour formula, determine the total electromagnetic cross section for light electron scattering. This is known as the Thomson cross section:

$$\sigma_T = \frac{8\pi}{3} r_e^2 \tag{32}$$

Evaluate the Thomson cross section numerically, without looking up any numbers.

(vii) Plot the polarization asymmetry

$$\frac{\frac{d\sigma_\parallel}{d\Omega} - \frac{d\sigma_\perp}{d\Omega}}{\frac{d\sigma_\parallel}{d\Omega} + \frac{d\sigma_\perp}{d\Omega}} \tag{33}$$

as a function of scattering angle $\theta$.

## Problem 6. Scattering from a perfectly conducting sphere

Consider light of wavenumber $k$ scattering off a perfectly conducting sphere of radius $a$. Assume that $ka \ll 1$ and that the skin depth is much less than the size of the sphere The incident light propagates along the $z$-direction.

(i) **Optional** Show that the external field $\boldsymbol{E} = E_o e^{-i\omega t}\boldsymbol{\epsilon}_o$ and $\boldsymbol{H} = H_o e^{-i\omega t}\boldsymbol{n} \times \boldsymbol{\epsilon}_o$ induces a time dependent electric and magnetic dipole moment :

$$\boldsymbol{p} = 4\pi a^3 \boldsymbol{E}_o e^{-i\omega t} \qquad \boldsymbol{m} = -2\pi a^3 \boldsymbol{H}_o e^{-i\omega t} \tag{34}$$

For the magnetic case you can look at the solutions to homework 5 (pages 2-6). For the electric case you can look at lecture 3.

(ii) By computing the radiated power from the time dependent magnetic and electric dipole, show that for arbitrary initial polarization $\epsilon_o$ of the incoming light, the scattering cross section off the sphere, summed over outgoing polarizations is given by:

$$\frac{d\sigma}{d\Omega}(\boldsymbol{\epsilon}_o, \boldsymbol{n}_o, \boldsymbol{n}) = k^4 a^6 \left[\frac{5}{4} - |\boldsymbol{\epsilon}_o \cdot \boldsymbol{n}|^2 - \frac{1}{4}|\boldsymbol{n} \cdot (\boldsymbol{n}_o \times \boldsymbol{\epsilon}_o)|^2 - \boldsymbol{n}_o \cdot \boldsymbol{n}\right] \tag{35}$$

where $\boldsymbol{n}_o$ and $\boldsymbol{n}$ are the directions of the incident and scattered radiations, while $\boldsymbol{\epsilon}_o$ is the (perhaps complex) unit polarization vector of the incident radiation ($\boldsymbol{\epsilon}_o^* \cdot \boldsymbol{\epsilon}_o = 1$; $\boldsymbol{n}_o \cdot \boldsymbol{\epsilon}_o = 0$).

Hint: as an intermediate step in the calculation show that

$$\boldsymbol{E}_{\rm rad} = \frac{-\omega^2}{4\pi c^2}\frac{e^{-i\omega t + kr}}{r} D_o \left[-\epsilon_o + \boldsymbol{n}(\boldsymbol{n} \cdot \epsilon_o) - \tfrac{1}{2}\boldsymbol{n} \times (\boldsymbol{n}_o \times \epsilon_o)\right] \tag{36}$$

where $D_o = 4\pi a^3 E_o$. Then square this result (repeating to yourself like the the little engine . . . “I think I can, I think I can, think I can”) using the front cover of Jackson.

(iii) If the incident radiation is linearly polarized, show that the cross section is

$$\frac{d\sigma}{d\Omega}(\boldsymbol{\epsilon}_o, \boldsymbol{n}_o, \boldsymbol{n}) = k^4 a^6 \left[\frac{5}{8}(1 + \cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\phi\right] \tag{37}$$

where $\boldsymbol{n} \cdot \boldsymbol{n}_o = \cos\theta$ and the azimuthal angle $\phi$ is measured from the direction of the linear polarization.

(iv) What is the ratio of the scattered intensities at $\theta = \pi/2$, $\phi = 0$ and $\theta = \pi/2$, $\phi = \pi/2$? Explain physically in terms of the induced multipoles and their radiation patterns.

## Problem 7. (Optional) Estimates

Without looking up numbers make the following estimates[1]. Explain qualitatively how you arrived at your estimate from the Lienard-Wiechert potentials.

(i) The light source NSLS II at BNL circulates electrons at 3 GeV with a circumference of 792 m. (*i*) Estimate the energy lost per turn to radiation. (*ii*) Estimate the energy of the typical photon which is emitted, and compare this energy with the energy of the electron. (*iii*) Estimate the angular width of the radiation cone.

(ii) The LHC at CERN circulates protons at 7 TeV with a circumference of 27 km. (*i*) Estimate the energy lost per turn for a proton at the LHC. (*ii*) Estimate the energy a typical photon that is emitted at the LHC due to synchrotron radiation, and compare this to the proton energy. (*iii*) Estimate the angular width of the radiation cone.

[1] You really need to know these numbers to get through life:

$$\alpha = \frac{e^2}{4\pi\hbar c} \simeq \frac{1}{137} \qquad \hbar c = 197\,\mathrm{eV\,nm} \tag{38}$$

$$m_e c^2 = 0.511\,\mathrm{MeV}\ \text{(half an MeV)} \qquad m_p c^2 = 0.938\,\mathrm{MeV}\ \text{(2000 times the electron mass )} \tag{39}$$

Seriously. . . they wont be given on the final and you may need them, togethewith the Bohr model estimates.