Problem 1. Practice with delta-fcns

A delta-function is a infinitely narrow spike with unit integral. $\int dx\,\delta(x) = 1$.

(a) (Optional). A theta function (or step function) is

$$\theta(x - x_o) = \begin{cases} 1 & x > x_o \\ 0 & x < x_o \\ \frac{1}{2} & x = x_o \end{cases} \tag{1}$$

Not worrying about the case when $x = x_o$, show that

$$\frac{d}{dx}\theta(x - x_o) = \delta(x - x_o) \tag{2}$$

(b) (Optional) Show that

$$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{3}$$

(c) (Optional) Using the identity of part (b), show that

$$\delta(g(x)) = \sum_m \frac{1}{|g'(x_m)|}\delta(x - x_m) \qquad \text{where } g(x_m) = 0 \text{ and } g_m'(x_m) \neq 0 \tag{4}$$

(d) Show that

$$\int_0^\infty dx\,\delta(\cos(x))\,e^{-x} = \frac{1}{2\sinh(\pi/2)} \tag{5}$$

The delta function $\delta(x)$ should be thought of as sequence of functions $\delta_\epsilon(x)$ – known as a Dirac sequence – which becomes infinitely narrow and have integral one. For example, an infinitely narrow sequence of normalized Gaussians

$$\delta(x) = \lim_{\epsilon\to 0}\delta_\epsilon(x) = \lim_{\epsilon\to 0}\frac{1}{\sqrt{2\pi\epsilon^2}}e^{-\frac{x^2}{2\epsilon^2}}\,. \tag{6}$$

The important properties are

$$1 = \int dx\,\delta_\epsilon(x) \tag{7}$$

and the convolution property

$$f(x) = \lim_{\epsilon\to 0}\int dx_o f(x_o)\delta_\epsilon(x - x_o) \tag{8}$$

I will notate any Dirac sequence with $\delta_\epsilon(x)$.

Delta functions are perhaps best thought about in Fourier space. In particular think about Eq. (**??**) in Fourier space. At finite epsilon this reads

$$f(k) \simeq f(k)\delta_\epsilon(k)\,. \tag{9}$$

So the Fourier transform of a Dirac sequence $\delta_\epsilon(k)$ should be essentially one, except at large k where the function $f(k)$ is presumably small.

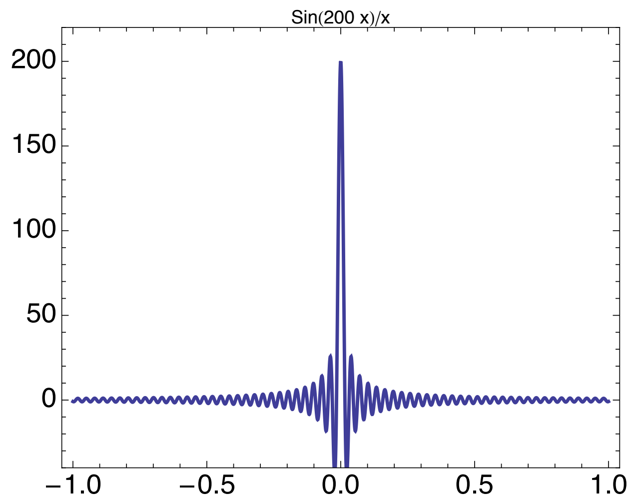
According to the uncertainty principle, a spike that has width $\Delta x \sim \epsilon$ in coordinate space, will have width $\Delta k \sim 1/\epsilon$ in k-space (momentum space). The meaningless formal expression

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} = \delta(x) \tag{10}$$

means that one should regulate this integral in some way and take the limit as the regulator ϵ goes to zero. For example, one could cut off the upper limit at a $k_{\rm max} = 1/\epsilon$,

$$\delta_\epsilon(x) = \int_{-1/\epsilon}^{1/\epsilon} \frac{dk}{2\pi}\, e^{ikx} = \frac{\sin(x/\epsilon)}{\pi x} \tag{11}$$

Making a graph of this function (with $1/\epsilon = 200$):



we see that for small ϵ it is infinitely narrow spike. Integrate around this spike between $-\Delta \ldots \Delta$, where Δ is small compared to one $\Delta \ll 1$, but much greater than ϵ, $\Delta \gg \epsilon$

$$I_\epsilon = \int_{-\Delta}^{\Delta} \mathrm{d}x\, \frac{\sin(x/\epsilon)}{\pi x} \tag{12}$$

$$= \int_{-\Delta/\epsilon}^{\Delta/\epsilon} \mathrm{d}u\, \frac{\sin(u)}{(\pi u)} \tag{13}$$

$$\simeq \int_{-\infty}^{\infty} \mathrm{d}u\, \frac{\sin(u)}{(\pi u)} \tag{14}$$

$$\simeq 1 \tag{15}$$

In the last steps we extended the integration to ∞ (since $\Delta/\epsilon \gg 1$), and have used the table integral, $\int_{-\infty}^{\infty} du\, \sin(u)/(\pi u) = 1$. The approximation becomes exact in the limit $\epsilon \to 0$, and thus we have shown that

$$\delta(x) = \lim_{\epsilon \to 0} \delta_\epsilon(x) = \lim_{\epsilon \to 0} \frac{\sin(x/\epsilon)}{\pi x} \tag{16}$$

is a Dirac sequence.

The precise way in which you regulate the Fourier integral is unimportant. The next problem regulates the Fourier integral in a particularly common way.

(a) Consider the Fourier transform pair $f(x)$ and $f(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{ikx} f(x)$. Note that

$$f(k=0) = \int_{-\infty}^{\infty} dx f(x) \tag{17}$$

Without using Mathematica, compute the following Fourier transform

$$\delta_\epsilon(x) \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} e^{-\epsilon|k|} \tag{18}$$

(You can check your algebra by explicitly checking that $\int dx\, \delta_\epsilon(x) = 1$ by direct integration – explain to yourself why one knows this integral must be unity before doing the integral).

Verify that

$$\lim_{\epsilon\to 0} \delta_\epsilon(x) = \delta(x) \tag{19}$$

i.e. that $\delta_\epsilon(k)$ is a Dirac sequence. This is another proof that

$$\delta(x) = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{ikx} e^{-\epsilon|x|} = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{ikx} \tag{20}$$

Problem 2. The electric stress tensor

Recall that the stress tensor is the force per area. The force per volume f^j is (minus) the divergence of the stress tensor (see class notes)

$$f^j = -\partial_i T^{ij} \tag{21}$$

This follows from the conservation law

$$\partial_t g^j + \partial_i T^{ij} = 0 \tag{22}$$

where g^j is the momentum per volume, and the basic notion that the force is the time derivative of the momentum.

The force per volume in electrostatics is

$$f^j = \rho E^j \tag{23}$$

This form must be the divergence of something. As you will show in this excercise

$$\rho E^j = -\partial_i T_E^{ij} \tag{24}$$

where

$$T_E^{ij} \equiv -E^i E^j + \tfrac{1}{2} E^2 \delta^{ij} \tag{25}$$

(a) (Optional) First write the electrostatic Maxwell equations $\nabla \cdot \boldsymbol{E} = \rho$ and $\nabla \times \boldsymbol{E} = 0$ using tensor notation, and explain why $\partial_i E_j = \partial_j E_i$.

(b) Within the limits of electrostatics, show that the electric force on a charged body is related to a surface integral of the (electric) stress tensor:

$$F^j = \int_V d^3\boldsymbol{r}\, \rho(\boldsymbol{r})\, E^j = -\int_S dS\, n_i T_E^{ij} \tag{26}$$

where $T_E^{ij} = -E^i E^j + \frac{1}{2} E^2 \delta^{ij}$, i.e. show that $\rho E^j = -\partial_i T_E^{ij}$

Problem 3. A stress tensor tutorial

Do not turn in the optional parts.

(a) (Optional) Consider a plane of charge with surface charge density σ, use the boundary conditions (i.e. Gauss Law) to show that the electric field on either side is $\sigma/2$

(b) (Optional) Consider an ideal infinite parallel plate capacitor with surface charge densities σ and $-\sigma$ respectively. Without using the stress tensor machinery, show that the force per area on each of the plates is $\sigma^2/2$

(c) (Optional) Consider a charged perfectly conducting solid object of any shape. Explain physically why the electric field is: (i) normal to the surface, (ii) zero on the inside, (iii) and equal to

$$\boldsymbol{E} = \sigma\, \boldsymbol{n} \qquad \text{or} \qquad E^i = \sigma\, n^i \tag{27}$$

(d) Without using the stress tensor machinery, show that the force per area on the walls of any metal surface is $\sigma^2/2$. (*Hint:* how large is the self field? Use part (a).)

The physics of the stress tensor is easy illustrated by knowing that the stress tensor of ideal gas is $T^{ij}_{\rm gas} = p\,\delta^{ij}$, where p is the pressure (force per area). Thus, if one considers a wall separating two gasses of left and right pressures p_L and p_R (*i.e.* the normal vector is[1], $n^j = \delta^{jx}$), then the net force per area on the wall is

$$n_i T^{ij}_L - n_i T^{ij}_R = (p_L - p_R)\, n^j \tag{28}$$

Note: that it is only the differences in the stress tensor which are physically important.

(e) (Optional) Recall that the net force on any object

$$F^j = -\oint dS\, n_i T^{ij}\,, \tag{29}$$

which we derived from the conservation law

$$\partial_t g^j + \partial_i T^{ij} = 0\,. \tag{30}$$

Deduce from this that the net force per area on a wall separating two regions is

$$n_i(T^{ij}_{\rm out} - T^{ij}_{\rm in})\,. \tag{31}$$

(f) Using the electric stress tensor $T^{ij}_E = -E^iE^j + \frac{1}{2}E^2\delta^{ij}$, show that the force per area on the surface of a charged metal object is

$$\text{force-per-area} = \frac{\sigma^2}{2} n^j \tag{32}$$

where $\boldsymbol{n}$ points from inside the metal to out.

[1] The notation is to confuse/educate you – I could have written $\boldsymbol{n} = (1,0,0)$ or $\boldsymbol{n} = \hat{\boldsymbol{x}}$.

(g) Now consider a charged and isolated parallel plate capacitor with charge per area $-\sigma$ and $+\sigma$ on the left and right plates (so that the normal is $n^j = \delta^{jx}$). A plane of charge with charge per area $\sigma/2$ lies halfway between the plates.

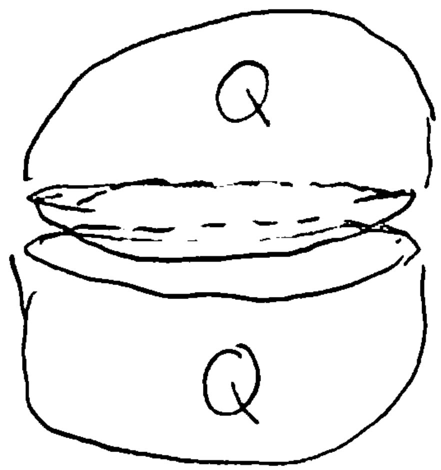
   (i) Compute all non-zero components of the stress tensor in the regions to the left and right of the plane of charge.

   (ii) Use the stress tensor to compute the force per area on the plane of charge, and show that it agrees with a simple minded approach.

Problem 4. Practice with the stress tensor

(a) Calculate the force between two (solid and insulating) uniformly charged hemispheres each with total charge Q and radius R that are separated by a small gap as shown below. You should find

$$F = \frac{3Q^2}{16\pi R^2} \tag{33}$$



Problem 5. Green function of a sphere

Consider a grounded, metallic, hollow spherical shell of radius R. A point charge of charge q is placed at a distance, a, from the center of the sphere along the z-axis. For simplicity take $a > R$.

(a) Start by momentarily setting $R = 1$, and therefore measure all lengths in units of R. a is then shorthand for a/R in this system of units. With these units, show that the distance from the point $\boldsymbol{r} = a\,\hat{\mathbf{z}}$ to any point, $\boldsymbol{n}$, on the surface of the sphere is equal (up to a constant factor of a) to the distance from a point at $\boldsymbol{r} = (1/a)\,\hat{\mathbf{z}}$ to the same point $\boldsymbol{n}$ on the sphere.

(b) Use the result of part (a) to construct the Green function of the grounded sphere of radius R using images, *i.e.* find the potential due to a point charge at $\boldsymbol{r} = a\hat{\mathbf{z}}$ in the presence of a grounded sphere.

(c) Compute the surface charge density, and show that it is correct by directly integrating to find the total induced charge on the sphere of part (b). You should find that the total induced charge is equal to the enclosed image charge (why?). Please do not use Mathematica to do integrals.

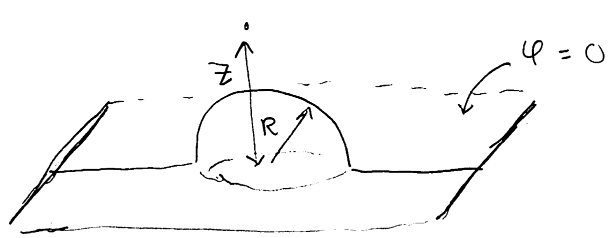
(d) Now consider a point charge of charge q at a point $\boldsymbol{r} = z\hat{\mathbf{z}}$ above a metallic hemisphere of radius R in contact with a grounded plane (see below). Determine the force on the charge as a function of z. You should find that at a distance $z = 2R$ the force is

$$F^z = -\frac{Q^2}{4\pi R^2}\left(\frac{737}{3600}\right) \tag{34}$$

(e) Show that at large distances, z, the Taylor series expansion for F^z is

$$F^z \simeq \frac{Q^2}{4\pi R^2}\left[\frac{-1}{4u^2} - \frac{4}{u^5} + \dots\right]$$

where $u = z/R$. Explicitly explain the coefficients of the series expansion (*i.e.* the $-1/4$ and -4) in terms of the multiple moments of the image solution.



Problem 6. An non-uniformly charged spherical shell

A hollow spherical shell of radius R is made of insulating material, and has a charge per unit area:

$$\sigma(\theta, \phi) = \sigma_o \left(\cos\theta + \tfrac{1}{2}\sin\theta\cos\phi\right) \tag{35}$$

(a) Find the potential for $r < R$ and $r > R$.

(b) From the asymptotics of your solution, determine the dipole moment $\boldsymbol{p}$ in Cartesian coordinates $\boldsymbol{p} = p_x\hat{\mathbf{x}} + p_y\hat{\mathbf{y}} + p_z\hat{\mathbf{z}}$.

(c) Determine the electric field inside the sphere in Cartesian coordinates.

Problem 7. Metal sphere in an electric Field

(a) A metal sphere of radius, a, lies in an electric field $\boldsymbol{E} = E_o \hat{\mathbf{z}}$. Determine the potential $\Phi(\boldsymbol{r})$ inside and outside of the sphere.

(b) Determine the induced surface charge density σ.

(c) By comparing the potental to the expectations of the multipole expansion, show that the induced dipole moment is

$$\boldsymbol{p} = 4\pi a^3 E_o \hat{\mathbf{z}} \tag{36}$$

You may check your work by integrating the induced charge density σ to find the dipole moment.