Problem 1. A rotating magnet

Consider a ring of radius a lying flat in the xy plane. Take the current in the ring to be $I(t) = I_0 \cos(\omega t)$.

(a) Determine the electric field close z axis for $z \gg a$ using the following two methods: (i) the Faraday Law relating the emf to the magnetic flux

$$\mathcal{E} = -\frac{1}{c}\partial_t \Phi_B \,, \tag{1}$$

and (ii) the vector potential. Work to lowest non-trivial order in $1/c$.

A magnetic dipole moment of magnitude m lying in the xy plane rotates about its center with angular velocity ω. It points in the x direction at time $t = 0$

(b) Find the electric and magnetic fields on the z axis as a function of time. Work to the lowest non-trivial order in inverse powers of c.

(c) Give a parametric estimate the magnitude of E/B at a given radius r (i.e. something of the form $E/B \sim$ (products of dimensionful parameters)) . At what radius is the solution you found in part (b) valid? At what radius does it break down and why?

Problem 2. A quick problem on the displacement current

A thin semi-infinite wire runs along the negative z axis, and carries a current $I(t) = I_o \cos(\omega t)$. The wire is abruptly terminated by a small metallic ball at the origin where the charge accumulates.

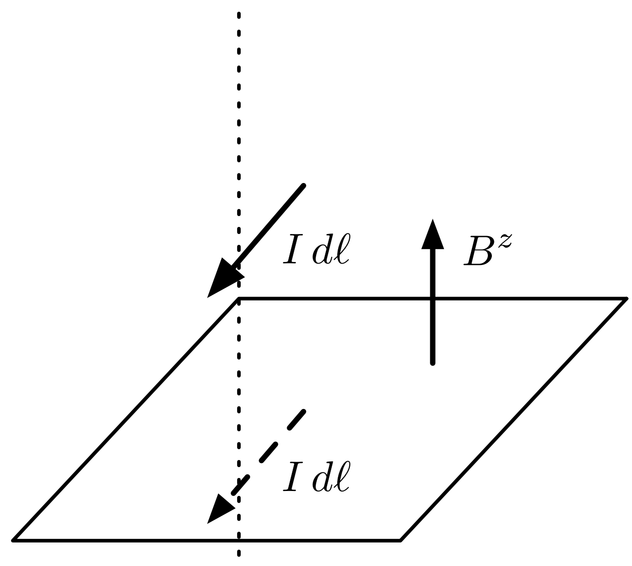
(a) Determine the magnetic field everywhere. (Assume that $\boldsymbol{B}$ points in the $\widehat{\boldsymbol{\phi}}$ direction.)

(b) Analyze your magnetic field from part (a) close to the z-axis for large negative z, and qualitatively explain the result.

(c) (Optional – do not hand in). Consider a current distribution which is mirror symmetric under reflections across the xy plane, i.e. under reflections $z \to -z$ accross the xy plane the current behaves as

$$J^x(x,y,-z) = J^x(x,y,z) \qquad J^y(x,y,-z) = J^y(x,y,z) \qquad J^z(x,y,-z) = -J^z(x,y,z)\,. \tag{2}$$

Show that the magnetic field in the xy plane is normal to the xy plane, i.e. only B^z is non-zero in the xy plane. (Hint break up the currents into $I\mathrm{d}\boldsymbol{\ell}$ where $I\mathrm{d}\boldsymbol{\ell}$ is either parallel perpendicular to the plane)



(i) Use this result to show that $\boldsymbol{B}$ points along the axis of an infinite solenoid.

(ii) Use this result to show that $\boldsymbol{B}$ points in the $\widehat{\boldsymbol{\phi}}$ direction for a torroidal magnet.

(iii) Use this result together with azimuthal symmetry to explain why $\boldsymbol{B}$ points in the $\widehat{\boldsymbol{\phi}}$ direction for part (a).

Problem 3. Electric and Magnetic fields of AC Solenoid

A cylindrical solenoid of high conductivity and radius a carries surface current $\mathbf{K} = K_o \cos(\omega t)\hat{\boldsymbol{\phi}}$

(a) Determine the electric and magnetic fields to the first non-vanishing order in the quasi-static approximation.

(b) Show that the magnetic field to the next-to-leading order in the quasi-static approximation outside the cylinder is

$$\Delta B = \delta B_z(\rho) - \delta B_z(\rho_{\max}) = \frac{K_o}{c} \cos(\omega t)\, \tfrac{1}{2}(\omega a/c)^2 \left(-\log\frac{\rho}{a} + C\right) \tag{3}$$

where C $= \log \rho_{\max}/a$. Here we are quoting ΔB the difference between δB at ρ and δB at $\rho_{\max}$.

(c) The cutoff $\rho_{\max}$ arises because the quasi static approximation breaks down for large ρ where the physics of radiation becomes important. $\rho_{\max}$ should be of order $\rho_{\max} \sim c/\omega$. Explain qualitatively why the approximation breaks down for this radius.

Remark: Certainly $|\delta B_z(\rho_{\max})|$ is logarithmically smaller than $|\delta B_z(\rho)|$ for $\rho \sim a$. In a logarithmic approximation we can neglect $\delta B_z(\rho_{\max})$ and set $\rho_{\max} = c/\omega$ leading to

$$\delta B_z(\rho) \simeq \frac{K_o}{c} \cos(\omega t)\, \tfrac{1}{2}(\omega a/c)^2 \left(-\log\frac{\rho}{a} + \log(c/(\omega a))\right) \tag{4}$$

Leading log accuracy may not be familiar to you. It just says that we are neglecting the constant inside the logarithm which is of order 1. Thus in this approximation,

$$\log(100/2) = \log(100) - \log(2) \simeq \log(100) \tag{5}$$

$$3.9 \simeq 4.6 \tag{6}$$

which is often good enough for government work. Bethe famously used such approximations to estimate the first QED corrections to the hydrodgen spectrum.

(d) Determine the magnetic field (to the next-to-leading order in the quasi-static approximation) inside the cylinder to logarithmic accuracy, and qualitatively sketch the complete magnetic field $B(\rho)/B_o$ where B_o is the leading order answer in the center of the cylinder.

Remark: Note that the ρ depence of part (b) and part (c) does not depend on the value of $C = \log(\rho_{\max}/a)$.

(e) Determine the vector and scalar potentials in the Coulomb and Lorentz gauges to the required order and accuracy to reproduce the electric and magnetic fields in part (a) and verify that you obtain the correct fields.

Problem 4. Eddy-Current Levitation (Zangwill)

A wire loop of radius b in the $x - y$ plane carries a time-harmonic current $I_o \cos \omega t$. Find the value of I_o needed to levitate a small sphere of mass m, radius a, and conductivity σ at a height z above the center of the loop. Assume $a \ll b$ and that $\delta \ll a$ where δ is the skin depth of the sphere.

Hints:

(a) First recall in class that we showed that an oscillating magnetic field is damped out over a distance δ. So the picture is that a surface current is generated to satisfy the boundary conditions:

$$\boldsymbol{n} \times (\boldsymbol{H}_{\rm out} - \boldsymbol{H}_{\rm in}) = \boldsymbol{K} \tag{7}$$

or

$$\boldsymbol{n} \times (\boldsymbol{H}_{\rm out}) = \boldsymbol{K} \tag{8}$$

since $\boldsymbol{H}_{\rm in} = 0$. In reality the "surface" current has a thin thickness of order δ, and integrating the current $\boldsymbol{j}$ over the thickness of order δ gives $\boldsymbol{K}$ (as you did in the inclass excercise).

(b) The boundary conditions satisfied by the magnetic field at the surface of the sphere are therefore

$$\boldsymbol{n} \cdot \boldsymbol{B}_{\rm out} = 0 \tag{9}$$

and

$$\boldsymbol{n} \times \boldsymbol{B}_{\rm out} = \boldsymbol{K} \tag{10}$$

(c) Thus, the strategy is to solve for the magnetic fields in the vacinity of the sphere (which are affected by the surface currents) with the boundary conditions given above, and the requirement that the field should asymptote to the field of the ring far from the sphere in units of the sphere radius, a. We are, however, still talking about distances very close to the sphere in units the ring radius b, i.e. for

$$a \ll r \ll b \tag{11}$$

the magnetic field approaches the magnetic field of the ring without the sphere at height z

This solution to the fields in the vacinity of the sphere will determine the surface current $\boldsymbol{K}$ and hence the induced magnetic moment on the sphere. I used the magnetic scalar potential, the solution for $\boldsymbol{B}$ resembles fluid flow around the sphere, and I find

$$\boldsymbol{m} = -\hat{\boldsymbol{z}}\,(2\pi a^3) B_o(z) \tag{12}$$

where $B_o(z)$ is the field from the ring at height z.

(d) Then you can compute the force by using the dipole moment and familiar formulas for forces on dipoles in external fields. I find that the time averaged force is

$$\overline{F_z} = \frac{3\pi}{4}\left(\frac{I_o}{c}\right)^2 \frac{a^3 b^4 z}{(z^2 + b^2)^4} = mg \tag{13}$$

Problem 5. Dipole down the tube (Zangwill)

A small magnet (weight w) falls under gravity down the center of an infinitely long, vertical, conducting tube of radius a, wall thickness $D \ll a$, and conductivity σ. Let the tube be concentric with the $z-$axis and model the magnet as a pointlike dipole with moment $\boldsymbol{m} = m\hat{\mathbf{z}}$. We can find the terminal velocity of the magnet by balancing its weight agains the magnetic drag force associated with the ohmic loss in the walls of the tube.

(a) At the moment it passes through $z = z_o(t)$, show that the magnetic flux produced by $\boldsymbol{m}$ through the a ring of radius a at height z' is

$$\Phi_B = \frac{m}{2}\frac{a^2}{r_o^3} \qquad \text{where} \qquad r_o^2 = a^2 + (z_o - z')^2 \tag{14}$$

(b) When the speed v of the dipole is small, argue that the Faraday EMF induced in the ring is

$$\mathcal{E} = -\frac{1}{c}\partial_t \Phi_B = \frac{v}{c}\frac{\partial \Phi_B}{\partial z'} \tag{15}$$

(c) Show that the current induced in the thin slice of tube which includes the ring is

$$dI = \frac{3mav\sigma D}{4\pi c}\frac{(z_o - z')}{r_o^5}dz' \tag{16}$$

(d) Compute the magnetic drag force $\mathbf{F}$ on $\boldsymbol{m}$ by equating the rate at which the force does work to the power dissipated in the walls of the tube by Joule heating. I find

$$\frac{dE_{\text{ohm}}}{dt} = \sigma D (v/c)^2 \frac{m^2}{a^4}\left(\frac{45}{1024}\right) \tag{17}$$

(e) Find the terminal velocity of the magnet.