Problem 1. Green theorem for first and second order equations and the initial value problem

First order: Consider a model first order equation equation for the velocity

$$m\frac{dv}{dt} + m\eta v = 0 \tag{1}$$

describing how a particle slows down.

(a) Determine the Green function for this equation, *i.e.* find the causal function that satisfies

$$\left[m\frac{d}{dt} + m\eta\right] G_R(t) = \delta(t) \tag{2}$$

using the direct method, and by fourier transforms.

(b) Show that for $t > t_o$

$$v(t) = mG_R(t, t_o)v(t_o) \tag{3}$$

is a solution to the differential equation, and satisfies the boundary conditions specified at $t = t_o$.

(c) Consider $G_R(t, t_o)$ as a function of t_o for fixed t. What equation and boundary conditions does $G_R(t, t_o)$ obey? Read the note on Green functions and linear operators.

Eq. (3) is normally how the Green function (propagator) is used in quantum mechanics. The Green function is used slightly differently for second order equations, since x and $\dot{x}$ enter the game.

Second order: In class we showed that the electrostatic potential can be determined from knowledge of the boundary value and the Dirichlet Green function. A very similar statement can be made about an initial value problem, *i.e.* the solution at future times can be determined from the initial conditions and the Green function.

For definiteness we will take a harmonic oscillator with mass m and resonant frequency ω_o:

$$m\frac{d^2x}{dt^2} + m\omega_o^2 x = 0\,.$$

The retarded Green function $G_R(t|t_o)$ is the position $x(t)$ of the harmonic oscillator at time t from an impulsive force at time t_o. It is causal, meaning that it vanishes whenever $t < t_o$, *i.e.*

$$\left(m\frac{d^2}{dt^2} + m\omega_o^2\right) G_R(t|t_o) = \delta(t - t_o) \qquad \text{and } G_R(t, t_o) = 0 \text{ for } t < t_o\ . \tag{4}$$

As always with Green functions, the second argument of the Green function obeys the adjoint equation, which in this (non-disaptive) case is the same algebraic equation but with advanced boundary conditions:

$$\left(m\frac{d^2}{dt_o^2} + m\omega_o^2\right) G_R(t|t_o) = \delta(t - t_o) \qquad \text{and } G_R(t, t_o) = 0 \text{ for } t_o > t\ . \tag{5}$$

(a) Given the initial conditions for the oscillator, $x(t_o)$ and $\partial_{t_o}x(t_o)$, at time t_o, show that the future value of the oscillator $x(t)$ is given by the Wronskian of the Green function and the initial conditions

$$x(t) = m\left[G_R(t,t_o)\partial_{t_o}x(t_o) - x(t_o)\partial_{t_o}G_R(t,t_o)\right] \qquad t > t_o \tag{6}$$

Do this in two ways:

(i) Read the notes online on Green fucntions and operators. Then prove Eq. (6) by starting with the statement that $\mathcal{L}^{\rm adj}_{t'}G_R(t,t') = \delta(t-t')$, and thus for $t > t_o$

$$x(t) = \int_{t_o}^{\infty} dt' x(t')\, \mathcal{L}^{\rm adj}_{t'}G_R(t,t') \tag{7}$$

(ii) Prove that Eq. (6) satisfies the equations of motion

$$\left(m\frac{d^2}{dt^2} + m\omega_o^2\right)x(t) = 0 \tag{8}$$

and the initial conditions,

$$\lim_{t\to t_o} x(t) = x(t_o) \tag{9}$$

$$\lim_{t\to t_o} \frac{dx(t)}{dt} = \partial_{t_o}x(t_o) \tag{10}$$

(b) Show that for the wave equation, $-\Box G_R(t\boldsymbol{x}|t_o\boldsymbol{x}_o) = \delta(t-t_o)\delta^3(\boldsymbol{x}-\boldsymbol{x}_o)$, the appropriate generalization of Eq. (6) is

$$u(t,\boldsymbol{x}) = \frac{1}{c^2}\int d^3\boldsymbol{x}_o \left[G_R(t\boldsymbol{x}|t_o\boldsymbol{x}_o)\partial_{t_o}u(t_o,\boldsymbol{x}_o) - u(t_o,\boldsymbol{x}_o)\partial_{t_o}G_R(t\boldsymbol{x}|t_o\boldsymbol{x}_o)\right] \tag{11}$$

where $u(t,\boldsymbol{x})$ satisfies the wave equation $-\Box u(t,\boldsymbol{x}) = 0$ together with the initial conditions specified by the function, $u(t_o,\boldsymbol{x})$, and its first derivative, $\partial_{t_o}u(t_o,\boldsymbol{x})$, at t_o. **Remark:** The results of this problem show that the general solution to the driven harmonic oscillator starting from some initial time moment t_o is

$$\frac{d^2x}{dt^2} + m\omega_o^2 x(t) = F(t) \tag{12}$$

is

$$x(t) = m\left[G_R(t,t_o)\partial_{t_o}x_o - x(t_o)\partial_{t_o}G_R(t,t_o)\right] + \int_{t_o}^{t} dt'\, G_R(t,t')F(t')\,. \tag{13}$$

At late times (in the presence of any infinitessimal damping) the initial conditions can be ignored.

Similarly for the first order equation:

$$\left[m\frac{d}{dt} + m\eta\right]v(t) = F(t)\,; \tag{14}$$

the general solution is

$$v(t) = mG_R(t,t_o)v(t_o) + \int_{t_o}^{t} dt'\, G_R(t,t')F(t')\,. \tag{15}$$

Problem 2. Green function of the Diffusion equation

Consider the homogeneous diffusion equation:

$$\partial_t n - D\nabla^2 n(t, \boldsymbol{r}) = 0\,. \tag{16}$$

The retarded Green function of the equation satisfies

$$\left[\partial_t - D\nabla^2\right] G(t\boldsymbol{r}|t_o\boldsymbol{r}_o) = \delta(t-t_o)\delta^3(\boldsymbol{r}-\boldsymbol{r}_o)\,. \tag{17}$$

with retarded boundary conditions.

(a) Write Eq. (17) in time and $\boldsymbol{k}$ by introducing the spatial Fourier transform

$$G(t, \boldsymbol{k}) \equiv \int d^3\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} G(t, \boldsymbol{r})\,, \tag{18}$$

and then determine the retarded Green function of the diffusion equation in $\boldsymbol{k}$ and time using the "direct" method.

(b) Determine the retarded Green function in ω and $\boldsymbol{k}$, $G_R(\omega, \boldsymbol{k})$, by Fourier transforming Eq. (17) in time and space. Verify that if you perform the Fourier integral over ω that you get the result of part (a).

(c) By taking the spatial Fourier transform verify that

$$G_R(\tau, \boldsymbol{r}) = \theta(\tau)\frac{1}{(2\pi\sigma^2(\tau))^{3/2}} \exp\left(-\frac{(\boldsymbol{r}-\boldsymbol{r}_o)^2}{2\sigma^2(\tau)}\right) \tag{19}$$

where $\sigma^2(t) = 2D\tau$ where $\tau = t - t_o$

Problem 3. Electric field in the far field

If you get stuck check the notes online. The scalar and vector potential in the far field are

$$\varphi(t, \boldsymbol{r}) = \frac{1}{4\pi r}\int d^3\boldsymbol{r}_o \rho(T, \boldsymbol{r}_o) \tag{20}$$

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{1}{4\pi r}\int d^3\boldsymbol{r}_o \boldsymbol{J}(T, \boldsymbol{r}_o)/c \tag{21}$$

where the retarded time $T = t - |\boldsymbol{r}-\boldsymbol{r}_o|/c$ in the far field is

$$T = t - r/c + \frac{\boldsymbol{n}\cdot\boldsymbol{r}_o}{c} \tag{22}$$

The goal is to compute the electric field

$$\boldsymbol{E}(t, r) = -\frac{1}{c}\partial_t \boldsymbol{A}(t, \boldsymbol{r}) - \nabla\varphi(t, \boldsymbol{r}) \tag{23}$$

(a) (Optional) Consider the change of variable $t, \boldsymbol{r}_o \to T, \boldsymbol{r}_o$. Show that

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial t} \tag{24}$$

$$\left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_T = \left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_t - \frac{\boldsymbol{n}}{c}\frac{\partial}{\partial t} \tag{25}$$

(b) Compute

$$\left(\frac{\partial}{\partial t} + c\boldsymbol{n}\frac{\partial}{\partial \boldsymbol{r}}\right) T \tag{26}$$

You should find a simple result. Interpret the answer using the definition of T

$T \equiv$ the time when the light should be emitted from $\boldsymbol{r}_o$ to arrive at the observation point $(t, \boldsymbol{r})$.

How do you interpert the derivative:

$$\left(\frac{\partial}{\partial t} + c\boldsymbol{n}\frac{\partial}{\partial \boldsymbol{r}}\right) \tag{27}$$

(c) (Optional) Show that

$$\boldsymbol{E} = -\frac{1}{4\pi r c^2}\int_{r_o} \frac{\partial \boldsymbol{J}(T, r_o)}{\partial T} + \frac{\boldsymbol{n}}{c}\frac{1}{4\pi r}\int_{r_o} \frac{\partial \rho(T, r_o)}{\partial T} \tag{28}$$

(d) (Optional) Use current conservation to express

$$\frac{\partial \rho(T, \boldsymbol{r}_o)}{\partial T} = -\left(\nabla_{r_o} \cdot \boldsymbol{J}\right)_T = -\left(\nabla_{r_o} \cdot \boldsymbol{J}\right)_t + \frac{\boldsymbol{n}}{c} \cdot \frac{\partial \boldsymbol{J}}{\partial T} \tag{29}$$

where $(\nabla_{r_o} \cdot \boldsymbol{J})_t$ denotes the divergence at fixed observation time

(e) (Optional) Conclude that only the transverse piece of the current to $\boldsymbol{n}$ contributes to the radiation field

$$\boldsymbol{E} = -\frac{1}{4\pi r}\frac{1}{c^2}\int_{\boldsymbol{r}_o} \underbrace{\left[\partial_t \boldsymbol{J} - \boldsymbol{n}(\boldsymbol{n} \cdot \partial_t \boldsymbol{J})\right]}_{\text{the part of } \partial_t J \text{ transverse to } \boldsymbol{n}} . \tag{30}$$

$$= \boldsymbol{n} \times \left[\frac{\boldsymbol{n}}{c} \times \frac{1}{4\pi r}\int_{\boldsymbol{r}_o} \frac{1}{c}\frac{\partial \boldsymbol{J}(T, r_o)}{\partial T}\right] \tag{31}$$

$$\tag{32}$$