Problem 1. A time dependent dipole

Consider an electric dipole at the spatial origin ($\mathbf{x} = 0$) with a time dependent electric dipole moment oriented along the z-axis, i.e.

$$\mathbf{p}(t) = p_0 \cos(\omega t) \mathbf{\hat{z}},$$

where $\mathbf{\hat{z}}$ is a unit vector in the z direction.

(a) Recall that the near and far fields of the time dependent dipole are qualitatively different. Estimate the length scale that separates the near and far fields.

(b) In the far field, how do the magnitude of the field strengths decrease with radius?

(c) Determine the ratio of $E/B$ at a distance $r$ in the far field.

(d) Estimate the total power radiated in a dipole approximation. How does this power depend on the dipole amplitude $p_0$, the oscillation frequency $\omega$, and the speed of light.

(e) In the near field regime, estimate how the electric and magnetic field strengths decrease with the radius $r$. ($r$ is the distance from the origin to the observation point.)

(f) Estimate the ratio $E/B$ at a distance $r$ in the near field. Is this ratio large or small?

(g) Determine the electric and magnetic fields to the lowest non-trivial order in the near field (or quasi-static) approximation.
Solution

1. The speed of light and the frequency define a length scale

\[ \frac{1}{R_o} = \frac{\omega}{c} \]

For distances less than \( R_o \) a quasi-static approximation may be used. For distances greater than \( R_o \) the finite speed of light must be considered to calculate the radiation fields.

2. In the far field both field strengths decrease as

\[ E \propto \frac{1}{r} \quad (2) \]

\[ B \propto \frac{1}{r} \quad (3) \]

3. The magnitudes are equal in a radiation field.

4. The power radiated from the Larmour is

\[ P \propto \frac{\omega^4}{c^3} p_o^2 \quad (4) \]

5. Standard dipole counting

\[ E \propto \frac{1}{r^3} \quad (5) \]

For the magnetic field

\[ \nabla \times B = \frac{1}{c} \partial_t E \quad (6) \]

suggests that

\[ B \propto \frac{\omega}{c r^2} \sim \frac{1}{R_o r^2} \quad (7) \]

6. Then, using the logic of the previous paragraph we see that

\[ \frac{E}{B} \sim \frac{R_o}{r} \gg 1 \quad (8) \]

7. There are various ways to do this. Perhaps the most direct is to use the gauge potentials in the lorentz gauge. We will not do this, but use the Maxwell equations directly.

The electric field in the near field region is just the field of a dipole

\[ E = \frac{1}{4\pi r^3} [3(p \cdot \hat{r})\hat{r} - p] \quad (9) \]

Clearly \( E \) lies in \( \hat{r}, \hat{\theta} \) plane. So

\[ E = \frac{1}{4\pi r^3} \left[ (2p_o(t) \cos \theta) \hat{r} + (p_o(t) \sin \theta) \hat{\theta} \right] \quad (10) \]
where \( p_o(t) = p_o \cos(\omega t) \)

Since

\[
\nabla \times B = \frac{1}{c} \partial_t E
\]

(11)

We try \( B \) in the \( \phi \) direction, with \( B_\phi(r, \theta) \). Then

\[
(\nabla \times B)_\theta = -\frac{1}{r} \partial_r(r B_\phi) = \frac{1}{4\pi r^3} (\partial_t p_o) \sin \theta
\]

(12)

Integrating with respect to \( r \) we find

\[
B_\phi = \frac{1}{4\pi r^2 c} (\partial_t p_o) \sin \theta + \frac{f(\theta)/R_o^2}{r}
\]

(13)

Where \( f(\theta) \) is a dimensionless integration constant, and we have inserted factors of \( R_o \) to make up the dimensions. The terms proportional to \( 1/r \) can be dropped in the near field regime since it is smaller by \( r/R_o \) than the \( \frac{1}{r^2} \) term. Thus

\[
B_\phi = \frac{1}{4\pi r^2 c} (\partial_t p_o) \sin \theta.
\]

(14)

Then one verifies that

\[
(\nabla \times B)_r = \frac{1}{r \sin \theta} \partial_\theta (\sin \theta B_\phi) = \frac{1}{4\pi r^3 c} (\partial_t p_o) 2 \cos \theta = \frac{1}{c} \partial_t E_r
\]

(15)

showing that \( B_\phi \) satisfies the Maxwell equations
Problem 2. Radiation in the lowest Bohr Orbit

In the Bohr model, a classical non-relativistic electron circles a proton in a circular orbit with angular momentum \( L = \hbar \), due to the Coulomb attraction between the electron and the proton.

(a) **(Optional)** Recall that that the electron kinetic energy is half of minus its potential energy (for a coulomb orbit). Recall also that the lowest bohr orbit has velocity, \( \beta = \alpha \) where \( \beta = v_e/c \), and \( \alpha = e^2 / (4\pi \hbar c) = 1/137 \). Prove these statements.

(b) Write down the (total=kinetic + potential) energy and radius of the lowest Bohr orbit in terms of the electron mass, \( m_e \), \( \hbar \), \( c \) and \( \alpha \). What is the size of the Bohr radius \( a_o \) compared to the electron compton wavelength, i.e. \( a_o/(\hbar/(m_e c)) \)?

(c) One of the difficulties with the Bohr model, is that classically the electron would radiate. Determine the energy lost to radiation per unit time, for an electron in the lowest orbit.

(d) Determine the energy radiated per revolution in the Bohr model, \( \Delta E \), and compare \( \Delta E \) to the (kinetic+potential) energy of the orbit, i.e. compute \( \Delta E / E_{\text{orbit}} \). Express \( \Delta E / E_{\text{orbit}} \) in terms of the fine structure constant, and estimate its value.

(e) If the electron moves in the \( x, y \) plane determine the time averaged power radiated per solid angle, \( \overline{dP}/d\Omega \).

You should find

\[
\frac{dP}{d\Omega} = \frac{e^2}{16\pi^2c^3} \frac{1}{2} (1 + \cos^2 \theta) (\omega_o^2 a_o)^2
\]  

where \( \omega_o \) is the angular velocity of the electron

(f) Check your result of part (e) by integrating over solid angle and comparing with part (c).

(g) Now we will study the polarization of the light. (These questions do not require calculation).

(i) If the emitted light is observed along \( x \) axis, what is the polarization of the radiated light? Explain physically.

(ii) If the emitted light is observed along the \( y \) axis, what is the polarization of the radiated light? Explain physically.

(iii) If the emitted light is observed along the \( z \) axis, what is the polarization of the light? Explain physically.

(h) The power radiated along the \( z \)-axis is twice as large as the power radiated along the \( x \)-axis. Explain this result physically.
a) Optional

For the Bohr orbit:

\[ F = \frac{m v^2}{r} \]

\[ \frac{e^2}{4\pi \epsilon_0 r^2} = \frac{m v^2}{r} \]

\[ e^2 = m v^2 \quad \Rightarrow \quad (-PE) = 2 \text{KE} \]

So,

\[ \frac{e^2}{4\pi \epsilon_0} = 2 \left( \frac{\hbar^2}{2ma_0^2} \right) \quad \text{the KE is} \quad \frac{\hbar}{2ma_0^2} \]

\[ a_0 = \frac{\hbar^2}{m (e^2 / 4\pi \epsilon_0)} = \frac{\hbar^2}{m c (e^2 / 4\pi \hbar c)} \]

\[ a_0 = \frac{\hbar^2}{mc \alpha} \]

\[ L = mvr \]

\[ \frac{\hbar}{ma_0} = V \quad \Rightarrow \quad V = \alpha c \]
Then

b) So

\[ E = KE + PE = -\frac{1}{2} PE + PE = \frac{PE}{2} \]

Then,

\[ E = -\frac{e^2}{2(4\pi\alpha)} = \frac{1}{2} mc^2 \alpha^2 \]

\[ a_o = \frac{h}{mc\alpha} \]

\[ \left( \frac{a_o}{\frac{h}{mc}} \right) = \frac{1}{\alpha} = 137 \]
Behr

a) \( a_0 = \frac{\hbar}{mc} \)

\[ E = -\frac{1}{2} \frac{e^2}{4\pi \alpha a_0} \]

b) Using the Larmour formula

\[ a = \omega^2 a_0 \]

\[ P = \frac{e^2}{4\pi} \frac{2}{3} \frac{a^2}{c^3} \]

\[ P = \frac{e^2}{4\pi} \frac{2}{3} \frac{\omega^4 a_0^2}{c^3} \]

\[ \omega_0 = \frac{\alpha c}{a_0} \]

\[ P = \frac{e^2}{4\pi} \frac{2}{3} \frac{a^4 c}{a_0^3} \]

\[ P = \frac{e^2}{4\pi \alpha_0^3} \frac{2}{3} \frac{a_0^4 c}{a_0} \]
c) So for part.

\[ \Delta E = \frac{2\pi a_0 P}{\alpha c} \]

So

\[ \Delta E = \frac{e^2}{4\pi a_0} \frac{2\pi a_0}{3} a_0 \frac{2\pi a_0}{c^2} \]

\[ \Delta E = \frac{e^2}{4\pi a_0} \frac{\pi a^3}{3} \]

And

\[ \Delta E \approx \frac{8\pi a^3}{3} \approx 10^{-6} \]

\[ E = \frac{8\pi a^3}{3} \]

\[ E \approx 10^{-6} \]

d) Using the Larmor result

\[ E_{rad} = \frac{e}{4\pi rc^2} \vec{n} \times \vec{n} \times a(t_e) \]

Using

\[ \vec{r} = (\cos \omega t_e, \sin \omega t_e, 0) a_0 \]

\[ \vec{a} = -\omega^2 a_0 (\hat{x} e^{i\omega t} + i\hat{y} e^{i\omega t}) \]

We will take

\[ \vec{a} = -\omega^2 a_0 e^{i\omega t} (\hat{x} + i\hat{y}) \]

the real part of this.
So then

\[
\frac{d\overline{P}}{d\Omega} = \frac{e^2}{16\pi c^3} \left| \hat{n} \times \hat{n} \times \hat{a} \right|^2
\]

\[
\hat{n} \times \hat{n} \times \hat{a} = -\hat{a} + \hat{n} (\hat{n} \cdot \hat{a})
\]

So taking \( \hat{n} \) in the \( x, z \) plane:

\[
\hat{n} \times \hat{n} \times \hat{a} \propto -(\hat{x} + i\hat{y}) + \hat{n} (\hat{n} \cdot (\hat{x} + i\hat{y}))
\]

\[
\propto -(\hat{x} + i\hat{y}) + \hat{n} (\hat{n} \cdot \hat{x})
\]

\[
\left| \hat{n} \times \hat{n} \times \hat{a} \right|^2 \propto (\hat{x}^2 + \hat{y}^2) - (\hat{n} \cdot \hat{x})^2
\]

\[
\propto 2 - \sin^2 \theta
\]

Then

\[
\frac{d\overline{P}}{d\Omega} = \frac{e^2}{16\pi c^3} \frac{1}{2} \left( 2 - \sin^2 \theta \right) (\omega_o^2 a_o)^3
\]
Problem 3. Radiation from a Phased Array

A current distribution consists of \( N \) identical sources. The \( k \)-th source is identical to the first source except for a rigid translation by an amount \( \mathbf{R}_k \) \((k = 1, 2, \ldots, N)\). The sources oscillate at the same frequency but have different phases \( \delta_k \). That is

\[
\mathbf{j}_k \propto \exp(-i(\omega t + \delta_k)) \tag{3}
\]

(a) Show that the angular distribution of radiated power can be written as a product of two factors: one is the angular distribution for \( N = 1 \); the other depends on \( \mathbf{R}_k \) and \( \delta_k \), but not on the structure of the sources.

(b) The planes of two square loops (each with sided length \( a \)) are centered on (and lie perpendicular to) the \( z \)-axis at \( z = \pm a/2 \). The loop edges are parallel to the \( x \) and \( y \) coordinate axes. Find the angular distribution of power in the \( x-z \) plane if the current at all points in both loops is \( I \cos(\omega t) \). Make a polar plot of the angular distribution of power for \( \omega c/a = 2\pi \) and \( \omega c/a \ll 1 \). Identify the multipole character of the radiation in the limit \( \omega a/c \ll 1 \).

You should find

\[
\frac{dP}{d\Omega} = \frac{I^2 a^2 \omega^2}{32 \pi^2 c^3} (2 \sin(\sin(\theta k a/2))^2 (2 \cos(\cos(\theta k a/2))^2 \tag{4}
\]

(c) The limit \( \omega a/c \ll 1 \) has a simple physical interpretation. Describe this interpretation and show that it reproduces all aspects of the power distribution (including normalization factors) in the limit \( \omega a/c \ll 1 \).

(d) Repeat part (b) when the current in the upper loop is \( I \cos \omega t \) and the current in the lower loop is \(-I \cos \omega t \).
Radiation from a phased Array

\[ \mathbf{A}_{\text{rad}} = \frac{1}{4\pi} \int_{\mathbf{r}_0}^{\mathbf{r}} \frac{\mathbf{I}}{c} \]  

The integration over the current is as follows:

- For each element we write
  \[ \mathbf{I}_o = \mathbf{I}_o + \Delta \mathbf{I} \]

- \[ \mathbf{J}_{o} (T, \Delta \mathbf{r}) = e^{i \omega (t - \frac{r}{c} + \frac{\pi}{c} R_0)} e^{-i \Delta \mathbf{r}} \]

- \[ \mathbf{J}_{o} (\Delta \mathbf{r}) = e^{i \omega (t - \frac{r}{c})} \frac{1}{e^{\frac{\pi}{c} R_0}} e^{-i \Delta \mathbf{r}} \]

\[ \mathbf{R}_o \] points from origin to center of square in Jth loop

\[ \mathbf{J} (\Delta \mathbf{r}) = \left( e^{-i \omega (t - \frac{r}{c})} \right) \frac{1}{e^{\frac{\pi}{c} R_0}} e^{-i \Delta \mathbf{r}} \]

So,

\[ \mathbf{A}_{\text{rad}} = e^{i \omega (t - \frac{r}{c})} \sum_{\Delta \mathbf{r}} e^{-i \omega n \cdot \Delta \mathbf{r} / c} \mathbf{J} (\Delta \mathbf{r}) e^{i \omega n \cdot R_0} e^{-i \Delta \mathbf{r}} \]

\[ = \left[ e^{i \omega (t - \frac{r}{c})} \sum_{\Delta \mathbf{r}} e^{-i \omega n \cdot \Delta \mathbf{r} / c} \mathbf{J} (\Delta \mathbf{r}) \right] \left[ \sum_{\Delta \mathbf{r}} e^{i \omega n \cdot R_0} e^{-i \Delta \mathbf{r}} \right] \]

\[ = \]
Phased Array Pg. 2

So taking the Poynting flux we have

\[
\frac{dP}{d\Omega} = \frac{r^2 \epsilon}{3} \left( \frac{\hat{n} \times \hat{n} \times \hat{r} \times \hat{r} \times \hat{A}_{\text{rad}}}{c^6} \right)^2
\]

\[\vec{E}_{\text{rad}}\]

We have since \(e^{-i\omega(t - r/c)}\) is an irrelevant overall phase

\[
\frac{dP}{d\Omega} = \frac{1}{c^3} \frac{-i\omega}{4\pi} \int \hat{n} \times \hat{n} \times g(\Delta r) e^{-i\omega \hat{n} \cdot \Delta r/c} \, d\Omega^0
\]

\[\sum_{\alpha} e^{-i\hat{d}_{\alpha} \cdot \hat{r}} e^{-i\omega \hat{n} \cdot \hat{r}_0/c} \]

\[\alpha\]
b) To work out the power from the array we must work out the integral

\[ \tilde{A} = \frac{1}{4\pi} \int \frac{e^{-i\omega \cdot \hat{n} \cdot \Delta \tau / c}}{\Delta \tau} \, \text{d}^3 \tau \]

from a single square loop. Taking \( \hat{n} \) in the \( x,z \) plane \( \hat{n} = (\sin \theta, 0, \cos \theta) \) as required by the problem statement.

3D view

Top view

\[ e^{-i\omega \hat{n} \cdot \Delta \tau / c} \] is with \( \Delta \tau = (x, -\frac{a}{2}, 0) \)

\[ e^{-i\omega \sin \theta x} \]

(3) Then the phase in leg 3 is also

\[ e^{-i\omega \sin \theta x} = e^{-i\omega \sin \theta x} \]
Since the phase is the same but the current is opposite

\[ \oint_{\mathbf{A}} d^3\mathbf{r} \ e^{-i\omega \mathbf{n} \cdot \mathbf{r}/c} \mathbf{j}(\mathbf{r}) = 0 \]

For legs 2 and 4 we have with \( \mathbf{r}_{2} = (x, y, 0) \)

\[ e^{-i\omega \mathbf{n} \cdot \mathbf{r}_{2}/c} = e^{-i\omega \sin \theta \mathbf{a}_{z}/2} \]

And for 4 \( \mathbf{r}_{4} = (-a_{z}, y, 0) \)

\[ e^{-i\omega \mathbf{n} \cdot \mathbf{r}_{4}/c} = e^{i\omega \sin \theta \mathbf{a}_{z}/2} \]

So

\[ \oint_{\mathbf{A}} d^3\mathbf{r} \ e^{-i\omega \mathbf{n} \cdot \mathbf{r}/c} \mathbf{j} = \int_{\mathbf{y}} \left( e^{-i\omega \sin \theta \mathbf{a}_{z}/2} - e^{i\omega \sin \theta \mathbf{a}_{z}/2} \right) \mathbf{I}_{0} \mathbf{y} \]

\[ = \mathbf{I}_{0} a \omega 2i \sin \left( \frac{\omega \sin \theta \mathbf{a}_{z}}{2} \right) \mathbf{y} \]

And thus

\[ \mathbf{r} \mathbf{A}_{1} = \frac{1}{2\pi} \mathbf{I}_{0} a \sin \left( \frac{\omega \sin \theta \mathbf{a}_{z}}{2} \right) \mathbf{y} \]

we need

\[ \mathbf{n} \times \mathbf{n} \times (\mathbf{r} \mathbf{A}_{1}) = -\mathbf{r} \mathbf{A}_{1} + \mathbf{n} \left( n \cdot \mathbf{r} \mathbf{A}_{1} \right) \]

\[ = -\mathbf{r} \mathbf{A}_{1} \]

\[ = -\mathbf{r} \mathbf{A}_{1} \]
To compute the power we have

\[
\frac{dP}{d\Omega} = \frac{1}{c^3} \frac{\omega^2}{2\pi} \left( \frac{I_0 a \sin(\omega \sin \theta \alpha)}{c} \right)^2 \left| \frac{1}{R_1} e^{-i\theta} + \frac{1}{R_2} e^{i\theta} \right|^2
\]

structure factor

So we have to work out the structure factor:

S.F. = \left| e^{-i\omega \cos \theta \alpha/c} + e^{i\omega \cos \theta \alpha/c} \right|^2 \quad \bar{R}_1 = (0, 0, \alpha_z) \quad \text{position of centers}

\bar{R}_2 = (0, 0, -\alpha_z)

\text{phases}

So the full result is

\[
\frac{dP}{d\Omega} = \frac{I_0^2 a^2 \omega^2}{16 \pi^2 c^3} \left( 2 \sin(\sin \theta \alpha z) \right)^2 \left( 2 \cos(\cos \theta \alpha z) \right)^2
\]

For \( \alpha z < 1 \) the second factor is 4, but the first factor is expanded

\[
\frac{dP}{d\Omega} = \frac{I_0^2 a^2 \omega^2}{16 \pi^2 c^3} \quad 4 \sin^2 \theta \left( \frac{\omega a}{c} \right)^2 \cdot 4
\]

\[
\frac{dP}{d\Omega} = \left( 2 \frac{I_0}{\alpha} a^2 \right)^2 \frac{1}{16 \pi^2 c^3} \sin^2 \theta \omega^4
\]

\( \text{this is the formula for magnetic dipole radiation} \)

\( \text{with} \quad \frac{\omega}{a} = 2 \frac{I_0}{\alpha} a^2 \quad \text{with the 2, because there are two antennas} \)
c) If the sign of the lower current is flipped then the structure factor changes

\[ S.F. = \left| e^{-i\omega \cos \theta a/2} - e^{-i\omega \cos \theta a/2} \right|^2 \]

\[ = \left( 2 \sin \left( \frac{\cos \theta \omega a}{2c} \right) \right)^2 \]

\[ \delta_1 = 0, \quad \delta_2 = \pi \]

\[ R_1, R_2 \text{ Same} \]

And

\[ \frac{dP}{d\Omega} = \frac{I_0^2 a^2 \omega^2}{16 \pi^2 c^3} \left( 2 \sin \left( \sin \theta \kappa a/2 \right) \right)^2 \left( 2 \sin \left( \cos \theta \kappa a/2 \right) \right)^2 \]

For small frequency we find magnetic quadrupole radiation:

\[ \frac{dP}{d\Omega} = \frac{e^2 (I_0^2)}{16 \pi^2 c} \frac{(\omega a)^6}{(c^2)} \sin^2 \theta \cos^2 \theta \]

Or

\[ \star \star \star \star \star \]
To check this result we integrate over $S_2$

$$\mathbf{\overline{P}} = \frac{e^2}{16\pi \epsilon_0^2} \left( \frac{\omega_o^2 a_o}{2} \right)^2 \int d\Omega \left( 2 - \sin^2 \theta \right)$$

Evaluating $I$:

$$I = 2\pi \int_{-1}^{1} dx \left( 2 - (1-x^2) \right) \quad x = \cos \theta$$

$$= 2\pi \int_{-1}^{1} dx \left( 1 + x^2 \right)$$

$$I = 2\pi \left( 2 + \frac{2}{3} \right) = \frac{16\pi}{3}$$

So, $$\mathbf{\overline{P}} = \frac{e^2}{16\pi \epsilon_0^2} \left( \frac{\omega_o^2 a_o}{2} \right)^2 \cdot \frac{16\pi}{3}$$

$$\mathbf{\overline{P}} = \frac{e^2}{6\pi \epsilon_0^2} \left( \frac{\omega_o^2 a_o}{2} \right)^2$$ which agrees with $\mathbf{A}$ from part b
Then the polarization is recorded by \( \hat{a}_\perp \)

\[
\hat{E}_{rad} = \frac{\hat{n} \times \hat{n} \times \alpha(t_e)}{4\pi rc^2} = \frac{\hat{\alpha}}{4\pi rc^2} [\dot{\alpha}(t)]
\]

Where

\[
\hat{\alpha} = -w^2 a_0 e^{-iwt_e} (\hat{x} + i\hat{y})
\]

- On the \( x \)-axis, \( \hat{a}_\perp = -w^2 a_0 e^{-iwt_e} (i\hat{y}) \), so \( \hat{E} \) points on the \( y \)-axis.

- On the \( y \)-axis, \( \hat{a}_\perp = -w^2 a_0 e^{iwt_e} (\hat{x}) \), so the polarization points on the \( \hat{x} \)-axis.

- On the \( z \)-axis, \( \hat{a}_\perp \propto e^{-iwt_e} (\hat{x} + i\hat{y}) \), which is circular like direction of motion.

\[
\text{Re} \ \hat{a}_\perp = (\cos wt) \hat{x} + (\sin wt) \hat{y} = \text{right circular}
\]
f) One can understand circular motion as a super-position of an x-oriented dipole and y-oriented dipole, 90° out of phase. On the x-axis only the radiation from the y-oriented dipole contributes as the x-oriented dipole is parallel to the observation direction (only transverse currents contribute to the radiation). On the z-axis both the x-oriented dipole and the y-oriented dipole contribute to the radiation field.

The two dipoles add incoherently.
Problem 4. A Charged Rotor: Zangwill

Two identical point charges of charge $q$ are fixed to the ends of a rod of length $2\ell$ which rotates with constant angular velocity of $\frac{1}{2}\omega$ in the $x - y$ plane about an axis perpendicular to the rod and through its center.

(a) Calculate the electric dipole moment $\mathbf{p}(t)$. Is there electric dipole radiation?

(b) Calculate the magnetic dipole moment $\mathbf{m}(t)$. Is there magnetic dipole radiation?

(c) Show that the electric quadruple moment is

$$ Q(t) = 3q\ell^2 \left( \begin{array}{ccc} \frac{1}{3} + \cos \omega t & \sin \omega t & 0 \\ \sin \omega t & \frac{1}{3} - \cos \omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{array} \right) $$

(d) Show that the time averaged angular distribution of radiated power is

$$ \frac{dP}{d\Omega} = \frac{1}{128\pi^2 c^5} \omega^6 q^2 \ell^4 (1 - \cos^4 \theta). $$
 Charged Rotor

Then consider

\[ \mathbf{m} = \frac{1}{2c} \int \mathbf{d}^3 x \times \mathbf{j} \]

\[ = \frac{1}{2c} \left[ \omega q l^2 \hat{z} + \omega q l^2 \hat{z} \right] = \frac{\omega q l^2 \hat{z}}{c} \]

But the magnetic moment is constant and does not contribute to the radiation.
2) Charged Rotor

(c) To find the quadrupole moment, we note the definition

\[ Q^{xy} = \int d^3r \, \rho(\mathbf{r}) \left( 3r^1 r^2 - r^3 \delta^{(3)} \right) \]

Then let's find the \( Q \) at this time \( t=0 \) and then rotate

\[ \begin{array}{c}
  \begin{array}{c}
    \cdot  \\
    \cdot  \\
    0
  \end{array}
  \\
  \begin{array}{c}
    \cdot  \\
    \cdot  \\
    \cdot
  \end{array}
\end{array} \]

\( Q^{xy} \) is clearly zero for the off diagonal components. But the diagonal components are

\[ Q^{xx} = \left[ (3 \, l^2 - l^2) + 3 \, (-l)^2 - (-l^2) \right] q_f \]

\[ Q^{xx} = 4q \, l^2 \]

\[ Q^{yy} = -2q \, l^2 \]

\[ Q^{zz} = -2q \, l^2 \]

Now we know the components in the rotated coordinate system. We want the components in the unrotated system.
(3) Charged Rotor

The appropriate rotation matrix is found from the picture:

\[ \bar{e}_x = \cos \theta \, \bar{e}_x + \sin \theta \, \bar{e}_y \]

\[ R^i_j = (R)^i_j = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

So

\[ Q^i_{jm} = (R)^i_j \, (R)^j_m \, Q^m_n \]

Or in matrices

\[ Q = \bar{Q} \, Q \, \bar{Q}^T \]

\[ Q = q l^2 \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \times \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \times \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]
 Charged Rotor

Multiplying out and using

\[ 1 + 3 \cos 2\theta = 4 \cos^2 \theta - 2 \sin^2 \theta \]

gives with \( 2\theta = \omega t \)

\[
\mathbf{Q} = \frac{q \ell^2}{3} \begin{pmatrix}
1 + 3 \cos \omega t & 3 \sin \omega t & 0 \\
3 \sin \omega t & 1 - 3 \cos \omega t & 0 \\
0 & 0 & -1/2
\end{pmatrix}.
\]

Or,

\[
\mathbf{Q} = 3q \ell^2 \begin{pmatrix}
\frac{1}{3} + \cos \omega t & \sin \omega t & 0 \\
\sin \omega t & \frac{1}{3} - \cos \omega t & 0 \\
0 & 0 & -2/3
\end{pmatrix}.
\]

In terms of matrices:

\(\mathbf{Q} = 3q \ell^2 \left[e^{-i\omega t} \mathbf{\sigma}_z + i e^{-i\omega t} \mathbf{\sigma}_x\right] + \text{const}\)

Where \(\text{const} = \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix}\)

and

\(\mathbf{\sigma}_x, \mathbf{\sigma}_z\) are Pauli matrices. It is understood that we take the real part of Eq.\n
\[
\mathbf{\sigma}_x = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \mathbf{\sigma}_z = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
(5) Charged Rotor

c) To find the power,

then we use

\[ \vec{E}(t, \vec{r}) = -\frac{1}{24\pi \varepsilon_0 c^3} \left[ \vec{\sigma} \cdot \vec{n} - \vec{n} \left( \vec{n} \cdot \vec{Q} \cdot \vec{n} \right) \right] \]

Looking at the form of \( \vec{Q} \):

\[ \vec{Q} = 3 q l^2 \left[ e^{-i\omega t} \sigma_x + i \sigma_z \right] + \text{const} \]

we see that

\[ \vec{E}(t, \vec{r}) = -i \vec{E}_1(t, \vec{r}) + \vec{E}_2(t, \vec{r}) \]

where \( \vec{E}_1 \) and \( \vec{E}_2 \) are real:

\[ \begin{align*}
    \vec{E}_1(t, \vec{r}) &= +\frac{3q l^2}{24\pi \varepsilon_0 c^3} \omega^3 \left[ (\vec{\sigma} \cdot \vec{n}) - \vec{n} \left( \vec{n} \cdot \vec{\sigma} \cdot \vec{n} \right) \right] \\
    \vec{E}_2(t, \vec{r}) &= +\frac{3q l^2}{24\pi \varepsilon_0 c^3} \omega^3 \left[ (\vec{\sigma} \cdot \vec{n}) - \vec{n} \left( \vec{n} \cdot \vec{\sigma} \cdot \vec{n} \right) \right]
\end{align*} \]

So

\[ \frac{dP}{d\Sigma} = \frac{c}{2} \text{Re} \left[ \vec{E} \cdot \vec{E}^* \right] r^2 \]

\[ \frac{dP}{d\Sigma} = \frac{c}{2} \left( |\vec{E}_1|^2 + |\vec{E}_2|^2 \right) r^2 \]
So we evaluate the square of the bracketed terms \( I^2 \) in Eq. (*):

\[
\begin{align*}
&\left[ \sigma \cdot \vec{n} - \vec{n} (\vec{n}^T \sigma \vec{n}) \right]^T \left[ \sigma \cdot \vec{n} - \vec{n} (\vec{n}^T \sigma \vec{n}) \right] \\
= &\quad (n^T \sigma - (n^T \sigma n) n^T) (\sigma \cdot n - n (n^T \sigma n)) \\
= &\quad n^T \sigma^2 n - 2(n^T \sigma n)^2 + (n^T \sigma n)^2 n^T n \\
= &\quad n^T \sigma^2 n - (n^T \sigma n)^2.
\end{align*}
\]

We used that \( \sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) for any pauli matrix and \( n \cdot n = 1 \) for a unit vector.

Then we find

\[
\begin{align*}
(A) \quad \frac{dP}{d\sigma} = &\quad \frac{c}{2} \left( \frac{3g l^2 W^3}{24 \pi^2 c^3} \right)^2 \left[ 2(n^T \sigma^2 n) - (n^T \sigma_x n)^2 - (n^T \sigma_z n)^2 \right]
\end{align*}
\]

Taking \( \vec{n} \) in the \( x-z \) plane:

\[
\vec{n} = (\sin \theta, 0, \cos \theta)
\]

\[
\begin{align*}
n^T \sigma_x n &= (\sin \theta, 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\sin \theta) = 0 \\
n^T \sigma_z n &= (\sin \theta, 0) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\sin \theta) = \sin^2 \theta
\end{align*}
\]
and

\[ n^T \sigma^2 n = (\sin \theta \ 0 \ \cos \theta) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sin \theta \\ 0 \\ \cos \theta \end{pmatrix} = \sin^2 \theta \]

So the term in square brackets in Eq. 4 on previous page is

\[ \begin{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 \sin^2 \theta - (\sin^2 \theta)^2 \end{bmatrix} = \begin{bmatrix} 1 - \cos^4 \theta \end{bmatrix} \]

And restoring constants we find from Eq. 4

\[ \frac{d \Phi}{d \Omega} = \frac{9^2 \epsilon_0 \omega^6}{128 \pi^2 c^5} (1 - \cos^4 \theta) \]
Problem 5.  (Almost all optional) Basics of Relativity

Do not hand in optional parts

(a) (Optional) The space time event at $X^\mu = (X^0, X^i) = (ct, \mathbf{x})$ happens at $X'^\mu = (X'^0, X'^i) = (ct', \mathbf{x})$ according to an observer moving to the right along the $x$ axis with velocity $v$. Define the “light-cone” coordinates $x^+ \equiv X^0 + X^1$ and $x^- \equiv X^0 - X^1$. Show that under this boost that the $x^+$ coordinates are contracted, while the $x^-$ coordinates are elongated

$$x^+ = e^{-y}x^+ = \sqrt{\frac{1-\beta}{1+\beta}}x^+, \quad (7)$$

$$x^- = e^y x^- = \sqrt{\frac{1+\beta}{1-\beta}}x^- . \quad (8)$$

Here

$$y = \tanh^{-1} \beta = \frac{1}{2} \log \left( \frac{1+\beta}{1-\beta} \right) \quad (9)$$

is the so-called “rapidity” of the boost. What is $x^+ x^-$ and why is it unchanged under boost?

(b) (Optional) A Lorentz tensor transforms as

$$T^\mu_{\rho \sigma} = L^\mu_{\rho \sigma} T^\rho_{\sigma} \quad (10)$$

Show that the transformation rule can be alternatively written

$$T^\mu_{\nu} = (\mathcal{L})^\mu_{\rho} T^\rho_{\sigma} (\mathcal{L}^{-1})^\sigma_{\nu} \quad (11)$$

or equivalently

$$T^\mu_{\nu} = L^\mu_{\rho} L^\sigma_{\nu} T^\rho_{\sigma} \quad (12)$$

(c) (Optional) The frequency and wave number of a plane wave of light, $e^{-i\omega t + ik \cdot x} = e^{iK \cdot x}$, form a lightlike four vector

$$K^\mu = \left( \frac{\omega}{c}, \mathbf{k} \right) \quad (13)$$

(i) Show that $K \cdot K = K^\mu K^\mu = 0$ (this is the statement that $K$ is lightlike.)

(ii) If a photon has frequency $\omega_0$ and is propagating along the $z$-axis, show (using the 4-vector properties of $K^\mu$) that according to an observer propagating in the negative $z$ direction with speed $\beta$

$$\omega = \sqrt{\frac{1+\beta}{1-\beta}} \omega_0 \quad (14)$$

(d) (Optional) Show that the four velocity $U^\mu = dx^\mu / d\tau$ satisfies $U_\mu U^\mu = -c^2$. 

5
(e) (Optional) For a particle with four momentum $P^\mu = (\frac{E}{c}, \bm{p}) = mU^\mu$ show that $P_\mu P^\mu = -(mc^2)^2/c^2$. This determines $E(p)$ the relation between energy and momentum:

$$\frac{E(p)}{c} = \sqrt{\bm{p}^2 + (mc)^2}.$$  \hfill (15)

(i) Show the velocity of the particle (i.e. the group velocity) is

$$v_p = \frac{\partial E(p)}{\partial \bm{p}} = \frac{c^2 \bm{p}}{E}.$$  \hfill (16)

(f) (Do me! Not optional) A particle with velocity $v_p$ in the x direction. Using the 4-vector transformation properties of $U^\mu$, show that according to an observer moving to the right with velocity $v$, the particle moves with velocity

$$v_p = \frac{v_p - v}{1 - v_p v/c^2}.$$  \hfill (17)
Basics

The Lorentz transformation is

\[
\begin{pmatrix}
X^0 \\
X^1
\end{pmatrix} = \begin{pmatrix}
\gamma & \gamma \beta \\
-\gamma \beta & \gamma
\end{pmatrix}
\begin{pmatrix}
X'^0 \\
X'^1
\end{pmatrix}
\]

So

\[x^+ = x^0 + x^1\]

\[x^- = x^0 - x^1\]

And

\[x^+ = x^0 + x^1\]

\[= \gamma x^0 - \gamma \beta x^1 + \gamma \beta x^0 + \gamma x^1\]

\[= \gamma (x^0 + x^1) - \gamma \beta (x^0 + x^1)\]

\[= \gamma (1 - \beta) x^+\]

\[x^+ = \frac{\gamma}{\sqrt{1 + \beta}} x^+\]

Note

\[\tan \theta y = \beta \text{ or } y = \frac{1}{2} \log \left( \frac{1 + \beta}{1 - \beta} \right)\]
So

\[ x^+ = e^{-y} x^+ \]

Similarly

\[ x^- = x^0 - x^1 \]
\[ = \gamma (1 + \beta) x^- \]
\[ = \sqrt{1 + \beta} \gamma x^- \]
\[ x^- = e^{+y} x^- \]

b) \[ T^\nu_{\mu} = L^\mu_{\rho} L^\nu_{\sigma} T^\rho_{\sigma} \]

Then raising/lowering \( \sigma \) and lowering \( \nu \)

\[ T^\nu_{\mu} = L^\mu_{\rho} L^\nu_{\sigma} T^\rho_{\sigma} \]
\[ = (L^\mu)_{\rho} (L^{-1^\tau})_{\sigma} T^\rho_{\sigma} \]
\[ T^\nu_{\mu} = (L^\mu)_{\rho} T^\rho_{\sigma} (L^{-1})^\sigma_{\nu} \]
c) A plane wave of light is

\[ e^{-i \frac{\omega}{c} x^0 + i k x^1} = e^{i \phi} \quad (\star) \]

Under boost the speed of light is constant. The only way to guarantee this is if the phase is invariant under Lorentz transformations.

\[ K^m = \left( \frac{\omega}{c}, k \right) \]

Then

\[ K_\mu X^\mu = -\frac{\omega}{c} x^0 + k x^1 \]

So

i) \[ K_\mu K^\mu = -\left(\frac{\omega}{c}\right)^2 + k^2 = 0 \]

since \( \omega = c k \) is required in Eq. (\star).

ii) Under boost in negative \( z \)-direction

\[
\begin{pmatrix}
\frac{\omega}{c} \\
k
\end{pmatrix} = \begin{pmatrix}
\gamma + \gamma \beta \\
\gamma \beta 
\end{pmatrix} \begin{pmatrix}
\frac{\omega}{c} \\
k
\end{pmatrix}
\]

\[ \frac{\omega}{c} = \gamma \frac{\omega}{c} + \gamma \beta k \]
But \( k = \omega/c \) so

\[
\frac{\omega}{c} = \gamma (1+\beta) \frac{\omega}{c}
\]

\[
\frac{\omega}{c} = \sqrt{\frac{1+\beta}{1-\beta}} \frac{\omega}{c}
\]

d) \( U^\mu = \frac{dx^\mu}{d\tau} \) so

\[
U^\mu U_\mu = \frac{dx^\mu}{d\tau} \frac{dx^\mu}{d\tau} = \frac{(dx^\mu dx^\mu)}{(d\tau)^2} = -c^2 \frac{(d\tau)^2}{(d\tau)^2}
\]

\[
= -c^2
\]

We used the definition of proper time \( c^2 d\tau^2 = -dx^\mu dx^\mu = ds^2 \)

You can also use

\[
U^\mu = (\gamma c, \gamma v) = c (\gamma, \gamma \beta)
\]

So

\[
U^\mu U_\mu = c^2 \left[ -\gamma^2 + \gamma^2 \beta^2 \right] = -c^2
\]
e) For a particle at rest

\[ p^\mu = \left( \frac{E_0}{c}, 0, 0, 0 \right) \quad p^\mu p_\mu = -\frac{E_0^2}{c^2} \]

Boosting by a non-relativistic amount

\[
\left( \begin{array}{c}
E'c \\
p
\end{array} \right) = \left( \begin{array}{cc}
\gamma + \gamma \beta & \gamma \\
+ \gamma \beta & \gamma
\end{array} \right) \left( \begin{array}{c}
E_0 c \\
0
\end{array} \right)
\]

\[
\left( \begin{array}{c}
E'c \\
p
\end{array} \right) \approx \left( \begin{array}{cc}
1 + \beta & \beta \\
+ \beta & 1
\end{array} \right) \left( \begin{array}{c}
E_0 c \\
0
\end{array} \right) \quad \text{or} \quad p \approx \frac{E_0}{c} \beta
\]

Start:

- Particle at rest

\[ K \]

\[ \text{O person moving to \textbf{Left}} \]

Change frames so that particle is moving

\[ \text{O} \rightarrow \]

\[ K' \]

So the momentum in \( K' \) is to be

\[ p = \frac{E_0}{c} \beta = \text{demand it to be} \quad m'v \]
This fixes

\[ E_0 = mc^2 \]

Thus

\[ P \cdot P^\mu = \text{invariant} = - \frac{(mc^2)^2}{c^2} = -(mc)^2 \]

Thus in any frame

\[ -\frac{E^2}{c^2} + \frac{p^2}{c^2} = -(mc)^2 \]

or

\[ E = \sqrt{p^2 + (mc)^2} \]

Then

\[ v_p = \frac{\partial E(p)}{\partial p} = c \frac{p}{\sqrt{p^2 + (mc)^2}} = \frac{c^2 p}{E} \]

\[ f) \quad \begin{pmatrix} U^0 \\ U^1 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma \beta \\ -\gamma \beta & \gamma \end{pmatrix} \begin{pmatrix} U^0 \\ U^1 \end{pmatrix} \]

This is the transformation rule of \( U^\mu \)
\( \frac{v_p}{c} = \frac{U^1}{u^0} = -\frac{\gamma \beta u^0 + \gamma u^1}{\gamma u^0 - \gamma \beta u^1} \)

divide by \( \gamma u^0 \)

\( \frac{v_p}{c} = \frac{-\beta + v_p/c}{1 - \beta v_p/c} \)

\( \frac{v}{c} = \frac{v_p/c - \beta}{(1 - \beta v_p/c)} \)
Problem 6. One liners

(a) Starting from the Maxwell equations for $F^{\mu \nu}$ and the definition of $F^{\mu \nu}$, derive the wave equation $-\Box A^\mu = J^\mu / c$.

(b) Starting from the Maxwell equations for $F^{\mu \nu}$ in covariant form, show that we must have $\partial_\mu J^\mu = 0$ for consistency.

(c) (This is two lines) Show that the energy conservation and force laws

\begin{align*}
\frac{dE_p}{dt} &= qE \cdot v_p \quad (18) \\
\frac{dp}{dt} &= q(E + \frac{v_p}{c} \times B) \quad (19)
\end{align*}

can be written covariantly

\begin{equation}
\frac{dP_\mu}{d\tau} = F_{\mu \nu} u^\nu / c \quad (20)
\end{equation}

Note that $E_p$ (the energy of the particle) is different from $E$ the electric field.

(d) From Eq. (20) show that $P_\mu P^\mu$ is constant in time.

(e) Show that $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant under the gauge transform

\begin{equation}
A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(X) \quad (21)
\end{equation}

where $\Lambda$ is an arbitrary function of $X = (t, r)$.

(f) Given $F^{\mu \nu}$ the only two Lorentz invariant quantities are $F_{\mu \nu} F^{\mu \nu}$ and $F_{\mu \nu} \tilde{F}^{\mu \nu}$. Evaluate these two invariants in terms of $E$ and $B$.

\footnote{answers: $2(B^2 - E^2)$ and $-4E \cdot B$}
One liners

a) \(-\partial_\mu F^{\mu\nu} = J^\nu/c\)
\(-\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = J^\nu/c\)
\(-\left(\partial^\mu \partial^\nu\right) A^\mu + \partial^\nu \left(\partial^\mu A^\mu\right) = J^\nu/c\)

with \(\partial^\mu A^\mu = 0\) and \(D = \partial^\mu A^\mu\), find
\(-\Box A^\nu = J^\nu/c\)

b) \(\partial_\nu F^{\mu\nu} = J^\nu/c\)
\(\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu J^\nu/c\)

This
\(F^{\mu\nu}\) is anti-symmetric, while \(\partial_\mu A^\nu\) is symmetric, so this is zero
\(\partial_\nu J^\nu/c = 0\)

c) \(dE_p = q \dot{E} \cdot \dot{v}_p\)
\(d\vec{p} = q \left(\vec{E} + \frac{\vec{v}_p \times \vec{B}}{c}\right)\)
Then multiply both sides by $\gamma$. Use

$$\gamma \frac{d}{dt} = \frac{d}{\tau} = \frac{d}{\tau}$$

So that with $u^m = (\gamma c, \gamma v)$ and $E^i = F_{0i}$,

$$d \left( \frac{E_i}{c} \right) = \frac{q}{\gamma} F_{0i} \frac{u_i}{c}$$

$$d \left( \frac{p^i}{c} \right) = \frac{q}{\gamma} \left( E^i \gamma + \varepsilon_{ijk} \frac{u_j}{c} B^k \right)$$

$$= \frac{q}{\gamma} \left( F^i - \frac{u^0}{c} F^0 + F^i \frac{u^i}{c} \right)$$

So

$$\frac{d p^m}{dt} = \frac{q}{\gamma} F^m \frac{u^m}{c}$$

d)

$$P^m \frac{dp^m}{dt} = \frac{q}{\gamma} F_{0v} P^m u^v = \frac{q m}{c} F_{0v} u^v u^v = 0$$

Since $F_{0v}$ is antisymmetric, while

$u^m u^m$ is symmetric.
e) \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \)

So \( A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \lambda \)

\( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \partial_\mu \partial_\nu \lambda + \partial_\nu \partial_\mu \lambda \)

f) \( F_{\mu\nu} F^{\mu\nu} = -2E^2 + 2B^2 = -2(E^2 - B^2) \)

\[
F_{\mu\nu} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}
\]

\[
F_{\mu\nu} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}
\]

\[
F_{\mu\nu} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}
\]

\( F_{\mu\nu} F^{\mu\nu} = -4 \vec{E} \cdot \vec{B} \)