Maxwell Equations + Induction + Energy in Mag fields

\[ \nabla \cdot E = \rho_{\text{mat}} + P_{\text{ext}} \]

\[ \nabla \times B = \frac{j_{\text{mat}}}{c} + \frac{j_{\text{ext}}}{c} + \frac{1}{c^2} \frac{\partial E}{\partial t} \]

\[ \nabla \cdot B = 0 \]

\[ -\nabla \times E = \frac{1}{c^2} \frac{\partial B}{\partial t} \]

Then, for the material current we write

\[ j_{\text{mat}} = \frac{\sigma}{c^2} \frac{\nabla \times H}{c} + \frac{1}{c} \frac{\partial P}{\partial t} + \nabla \times M \]

Then with continuity, \( \rho_{\text{mat}} = -\nabla \cdot P, \quad D = E + P, \quad H = \vec{B} - \vec{M} \), find

\[ \nabla \cdot D = P_{\text{ext}} \quad E + P \]

\[ \nabla \times H = \frac{j_{\text{ext}}}{c} + \frac{1}{c^2} \frac{\partial D}{\partial t} \quad \text{← maxwell eqs in simple matter} \]

\[ \nabla \cdot B = 0 \]

\[ -\nabla \times E = \frac{1}{c^2} \frac{\partial B}{\partial t} \]
Then we expand in powers of $c$

**Electrostatics**

\[ \nabla \cdot D^{(0)} = \rho_{\text{ext}} \]
\[ \nabla \times E^{(0)} = 0 \]

**Magnetostatics**

\[ \nabla \times H^{(1)} = \frac{j_{\text{ext}}}{c} + \frac{1}{c} \frac{\partial D^{(0)}}{\partial t} \]
\[ \nabla \cdot B^{(1)} = 0 \]

**Induced Electric fields / Back Emf**

\[ \nabla \cdot D^{(2)} = 0 \]
\[ -\nabla \times E^{(2)} = \frac{1}{c} \frac{\partial B^{(1)}}{\partial t} \] could call $E^{(2)} = E^{\text{ind}}$

Want to compute the energy stored in magnetic field

Back Emf → $I$

Imagine slowly increasing the current changing the current makes a changing magnetic field inducing a Back Emf. The work the Battery does to increase the current is the energy stored in the fields
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almost

Take magnetostatics, i.e. \( D(\omega) = 0 \)

\[
\nabla \times H = -j
\]
\[
\nabla \cdot B = 0
\]

And

\[
- \nabla \times E^{\text{ind}} = \frac{1}{c} \frac{\partial J}{\partial t} \quad B
\]

Then the work by battery is

\[
\frac{\delta U}{\delta t} = \frac{\delta W_{\text{batt}}}{\delta t} - \int_{\Gamma} \frac{\nabla \cdot E^{\text{ind}}}{\delta t} \cdot \mathbf{v}
\]

\[
= - \int (\nabla \times H) \cdot \delta E^{\text{ind}}
\]

\[
\nabla \cdot (H \times E) = (\nabla \times H) \cdot E
\]

\[
- H \cdot \nabla \times E
\]

\[
= - \int \mathbf{H} \cdot \epsilon \nabla \times \delta E^{\text{ind}}
\]

\[
\frac{\delta U}{\delta t} = \oint_{\Gamma} \mathbf{H} \cdot \delta \mathbf{B}
\]

\[
\delta U = \oint \mathbf{H} \cdot \delta \mathbf{B}
\]
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Then for linear media $\delta B = \mu \delta H$

$$U = \frac{1}{2} \int \frac{\mathbf{H}^2}{\mu} = \frac{1}{2} \int \mathbf{H} \cdot \mathbf{B} \ d^3x = U$$

These equations are often expressed in terms of $\mathbf{J}$ and $\mathbf{A}$ rather than $\mathbf{B}$.

Indeed,

$$\delta U = \int_{V} \mathbf{H} \cdot \delta \mathbf{B}$$

$$\delta E_{\text{ind}} = -\frac{1}{c} \mathbf{A} \cdot \delta \mathbf{A}$$

$$\delta U = \int_{V} \nabla \times \mathbf{H} \cdot \delta \mathbf{A}$$

By parts (no minus) because cross prod

$$\delta U = \int_{\partial V} \frac{\partial \mathbf{A}}{\partial t} \cdot \delta \mathbf{A}$$

For linear media $\mathbf{SA} \propto \mu \delta \mathbf{J}$

$$U = \frac{1}{2} \int \frac{\mathbf{J} \cdot \mathbf{A}}{c}$$
Inductance in Wires

\( U_B = \frac{1}{2} \int \mathbf{j} \cdot \mathbf{A} \, d^3x = \frac{1}{2} \oint \mathbf{H} \cdot \mathbf{B} \)

\( U_B \) is a property of state

\( S \mathbf{U}_B = \oint \frac{\mathbf{j} \cdot d\mathbf{A}}{c} \)

For a set of wires: \( \oint \mathbf{j} \, d^3x = \int \mathbf{J} dl \)

Then find summed over \( n = \) loops

\( U = \frac{1}{2c} \oint \mathbf{I} \, \Phi_a \)

\( \Phi_a = \oint \mathbf{A} \cdot d\mathbf{l} = \oint \mathbf{B} \cdot d\mathbf{A} \)

\( S \mathbf{U} = \oint \frac{\Phi}{c} \)

Note that, \( \mathbf{A}(x) = \rho \int \frac{\delta(x-x_0)}{\sqrt{4\pi |x-x_0|}} \)

\( U_B = \frac{\mu}{2} \int d^3x \, d^3x', \frac{\mathbf{j}(x)}{c} \cdot \mathbf{j}(x')/c \)

\( \frac{4\pi}{(\pi x - x_0)} \)
So for a set of wires

\[ U = \frac{1}{2} M_{ab} I_a I_b \]

\[ M_{ii} \] is the self-inductance of the first loop

\[ M_{ip} \] is the mutual inductance between the 1st and 2nd

Then since \( U_B = \frac{1}{2} \frac{I_a}{c} \Phi_a \)

\[ \frac{\Phi_a}{c} = M_{ab} I_b \] \text{ back emf}

And for any circuit \( E_a = -\frac{1}{2} \frac{d}{dt} \Phi_B = -M_{ab} \frac{dI_b}{dt} \)
Problem on Mutual Inductance and Force

Compute the mutual inductance of a ring and a long straight wire.

\[ \Rightarrow \]

\[ \bigcirc \quad I_1 \]

\[ \Downarrow \quad D \]

\[ I_2 \]

Solution: Current in wire one

\[ U_{12} = \int_{c} \mathbf{A} \cdot d\mathbf{l}_2 \]

\[ = \frac{I_1}{c} \int_{c} \mathbf{A}_2 \cdot d\mathbf{l}_1 = \frac{I_1}{c} \int_{c} \mathbf{B}_2 \cdot d\mathbf{l}_1 \]

Then the field from the wire is

\[ \mathbf{B}_2 = \frac{I_2}{2\pi \rho} \]

So we need to integrate this field from the wire over the area of the ring.
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\[ \oint \mathbf{B} \cdot d\mathbf{a} = \frac{I_1 I_2}{C} 2 \left( R^2 - (p-D)^2 \right)^{1/2} \, dp \]

We have

\[ U_{12} = \frac{1}{2} \left( D - \sqrt{D^2 - R^2} \right) \]

So \[ M_{12} = \frac{1}{C^2} \left( D - \sqrt{D^2 - R^2} \right) \]

Then we might want to compute the force between the ring and the wire. To do this we ask about the change in \( U_{12} \) as the distance between the ring and the wire is changed:

\[ SU_{12} = I_a \Delta \Phi_a + 8W_{\text{mech}} \]

\[ \Delta \Phi_a = F \cdot SD \]

\[ 8W_{\text{batt}} \]

Stored in fields work done by battery to keep current fixed

\[ F = \frac{\Delta F_{\text{ring}}}{S} \]

force on ring applied mechanically
Mutual Inductance and Force pg. 3

\[ S_U = \frac{1}{2} I_a S m_{ab} I_b \]

\[ I_a s \Phi_a = I_a S m_{ab} I_b \]

So

\[ S U_B - I_a s \Phi_a = -\frac{1}{2} I_a S m_{ab} I_b = -F s \delta_D \]

So

\[ F^x = +S m_{ab} \frac{I_1 I_2}{S D} \quad \text{or} \quad \frac{I_1 I_2}{c^2} \left(1 - \frac{D}{\sqrt{D^2 - R^2}}\right) \]

\[ = -I_1 I_2 \left(\frac{D}{c^2} - 1\right) \frac{1}{\sqrt{D^2 - R^2}} \]

indicates an attractive force

i.e. force in negative x-direction

\[ \frac{F}{I_1 I_2 / c^2} \quad \leftarrow \text{attractive force} \]