

1 Introduction to the Maxwell Equations

1.1 The maxwell equations and units

- We use Heavyside Lorentz system of units. This is discussed in a separate note.
- The Maxwell equations are

$$\nabla \cdot \mathbf{E} = \rho \quad (1.1)$$

$$\nabla \times \mathbf{B} = \frac{\mathbf{j}}{c} + \frac{1}{c} \partial_t \mathbf{E} \quad (1.2)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.3)$$

$$-\nabla \times \mathbf{E} = \frac{1}{c} \partial_t \mathbf{B} \quad (1.4)$$

In integral form we have

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = Q_{\text{enc}} \quad (1.5)$$

$$\oint_\ell \mathbf{B} \cdot d\boldsymbol{\ell} = \frac{I}{c} + \frac{1}{c} \partial_t \Phi_E \quad (1.6)$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad (1.7)$$

$$-\oint_\ell \mathbf{E} \cdot d\boldsymbol{\ell} = \frac{1}{c} \partial_t \Phi_B \quad (1.8)$$

Here $\Phi_E = \int \mathbf{E} \cdot d\mathbf{S}$ is the electric flux, $\Phi_B = \int \mathbf{B} \cdot d\mathbf{S}$ is the magnetic flux, and $I = \int_S \mathbf{j} \cdot d\mathbf{S}$ is the current crossing a surface, S . $d\mathbf{S}$ is the surface element with a specified area and normal $d\mathbf{S} = \mathbf{n} d(\text{area})$. $d\boldsymbol{\ell}$ denotes a closed line integral element.

- We specify the currents and solve for the fields. In media we specify a constituent relation relating the current to the electric and magnetic fields.
- The Maxwell force law

$$\mathbf{F} = q \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \quad (1.9)$$

- Hemholtz Theorems state:

(a) Given a curl free vector field, $\mathbf{C}(\mathbf{x})$, there exists a scalar function, $S(\mathbf{x})$, such that $\mathbf{C} = -\nabla S$:

$$\text{if } \nabla \times \mathbf{C}(\mathbf{x}) = 0 \quad \text{then } \mathbf{C} = -\nabla S(\mathbf{x}) \quad (1.10)$$

(b) Given a divergence free vector field, $\mathbf{D}(\mathbf{x})$, there exists a vector field \mathbf{V} such that $\mathbf{D} = \nabla \times \mathbf{V}$:

$$\text{if } \nabla \cdot \mathbf{D}(\mathbf{x}) = 0 \quad \text{then } \mathbf{D} = \nabla \times \mathbf{V}(\mathbf{x}) \quad (1.11)$$

The converses are easily proved, $\nabla \times \nabla S(\mathbf{x}) = 0$, and $\nabla \cdot \nabla \times \mathbf{V}(\mathbf{x}) = 0$ There are two *very important* consequences for the Maxwell equations.

(a) From the source free Maxwell equations (eqs. three and four) one finds that

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (1.12)$$

$$\mathbf{E} = -\frac{1}{c} \partial_t \mathbf{A} - \nabla \phi \quad (1.13)$$

(b) Current conservation follows by manipulating the sourced maxwell equations (eqs. one and two)

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0 \quad (1.14)$$

- For a system of characteristic length L (say one meter) and characteristic time scale T (say one second), we can expand the fields in $1/c$ since $(L/T)/c \ll 1$:

$$\mathbf{E} = \mathbf{E}^{(0)} + \mathbf{E}^{(1)} + \mathbf{E}^{(2)} + \dots \quad (1.15)$$

$$\mathbf{B} = \mathbf{B}^{(0)} + \mathbf{B}^{(1)} + \mathbf{B}^{(2)} + \dots \quad (1.16)$$

where each term is smaller than the next by $(L/T)/c$. At zeroth order we have

$$\nabla \cdot \mathbf{E}^{(0)} = \rho \quad (1.17)$$

$$\nabla \times \mathbf{E}^{(0)} = 0 \quad (1.18)$$

$$\nabla \cdot \mathbf{B}^{(0)} = 0 \quad (1.19)$$

$$\nabla \times \mathbf{B}^{(0)} = 0 \quad (1.20)$$

These are the equations of electro statics. Note that $\mathbf{B}^{(0)} = 0$ to this order (for a field which is zero at infinity)

- At first order we have

$$\nabla \cdot \mathbf{E}^{(1)} = 0 \quad (1.21)$$

$$\nabla \times \mathbf{E}^{(1)} = 0 \quad (\text{since } \partial_t \mathbf{B}^{(0)} = 0) \quad (1.22)$$

$$\nabla \cdot \mathbf{B}^{(1)} = 0 \quad (1.23)$$

$$\nabla \times \mathbf{B}^{(1)} = \frac{\mathbf{j}}{c} + \frac{1}{c} \partial_t \mathbf{E}^{(0)} \quad (1.24)$$

This is the equation of magneto statics, with the contribution of the Maxwell term, $1/c \partial_t \mathbf{E}^{(0)}$, computed with electrostatics. Note that $\mathbf{E}^{(1)} = 0$